

# GLOBAL MILD SOLUTIONS OF FRACTIONAL NAIVER-STOKES EQUATIONS WITH SMALL INITIAL DATA IN CRITICAL BESOV-Q SPACES

PENGTAO LI, JIE XIAO, AND QIXIANG YANG

**ABSTRACT.** In this paper, we establish the global existence and uniqueness of a mild solution of the so-called fractional Navier-Stokes equations with a small initial data in the critical Besov-Q space covering many already known function spaces.

## CONTENTS

Statement of the main theorem	1
1. Meyer wavelets and Besov spaces	5
1.1. Meyer and Daubechies wavelets	5
1.2. Wavelet characterization of Besov spaces	6
2. Besov-Q spaces via wavelets	12
2.1. Definition and its wavelet formulation	12
2.2. Critical spaces and inclusion relations	19
3. Besov-Q spaces via semigroups	22
3.1. Wavelets and semigroups	22
3.2. Tent spaces generated by Besov-Q spaces	26
4. Non-linear terms and their a prior estimates	47
4.1. Decompositions of non-linear terms	47
4.2. Induced a prior estimates	52
5. Proof of the main theorem	60
6. Proof of Lemma 5.1	62
6.1. The setting (i)	62
6.2. The setting (ii)	67
6.3. The setting (iii)	72

---

2000 *Mathematics Subject Classification.* Primary 35Q30; 76D03; 42B35; 46E30.

*Key words and phrases.* FNS equations, Besov-Q spaces, mild solutions, existence, uniqueness.

PTL's research was supported by: NSFC No. 11171203 and No.11201280; New Teacher's Fund for Doctor Stations, Ministry of Education No.20114402120003; Guangdong Natural Science Foundation S2011040004131; Foundation for Distinguished Young Talents in Higher Education of Guangdong, China, LYM11063. JX was in part supported by NSERC of Canada and URP of Memorial University, Canada.

7.	Proof of Lemma 5.2	75
7.1.	The setting $1 < p \leq 2$	75
7.2.	The setting $2 < p < \infty$	87
8.	Proof of Lemma 5.3	95
8.1.	The setting $1 < p \leq 2$	95
8.2.	The setting $2 < p < \infty$	103
9.	Proof of Lemma 5.4	109
9.1.	The setting $1 < p \leq 2$	109
9.2.	The setting $2 < p < \infty$	115
10.	Proof of Lemma 5.5	119
10.1.	The setting $1 < p \leq 2$	119
10.2.	The setting $2 < p < \infty$	125
	References	130

## STATEMENT OF THE MAIN THEOREM

For  $\beta > 1/2$ , the Cauchy problem of the so-called Fractional Navier-Stokes (FNS) equations on the half-space  $\mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n, n \geq 2$ , is to decide the existence of a solution  $u$  to:

$$(0.1) \quad \begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^\beta u + u \cdot \nabla u - \nabla p = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ u|_{t=0} = a, & \text{in } \mathbb{R}^n, \end{cases}$$

where  $(-\Delta)^\beta$  represents the  $\beta$ -order Laplace operator defined by the Fourier transform in the space variable:

$$(-\Delta)^\beta u(\cdot, \xi) = |\xi|^{2\beta} \hat{u}(\cdot, \xi).$$

Upon letting  $R_j, j = 1, 2, \dots, n$ , be the Riesz transforms, writing

$$\begin{cases} \mathbb{P} = \{\delta_{ll'} + R_l R_{l'}\}, l, l' = 1, \dots, n; \\ \mathbb{P} \nabla(u \otimes u) = \sum_l \frac{\partial}{\partial x_l}(u_l u) - \sum_l \sum_{l'} R_l R_{l'} \nabla(u_l u_{l'}); \\ e^{-t(-\Delta)^\beta} f(\xi) = e^{-t|\xi|^{2\beta}} \hat{f}(\xi), \end{cases}$$

and using  $\nabla \cdot u = 0$ , we can see that a solution of the above Cauchy problem is then obtained via the integral equation:

$$(0.2) \quad \begin{cases} u(t, x) = e^{-t(-\Delta)^\beta} a(x) - B(u, u)(t, x); \\ B(u, u)(t, x) \equiv \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P} \nabla(u \otimes u) ds, \end{cases}$$

which can be solved by a fixed-point method whenever the convergence is suitably defined in a function space. Solutions of (0.2) are called mild

solutions of (0.1). The notion of such a mild solution was pioneered by Kato-Fujita [13] in 1960s. During the latest decades, many important results about the mild solutions to (0.1) (and its special case  $\beta = 1$ ) have been established; see for example, Lei-Lin [16], Cannone [3, 4], Germin-Pavlovic-Staffilani [10], Giga-Miyakawa [11], Kato [12], Koch-Tataru [15], Wu [30, 31, 32, 33], and Kato-Ponce [14].

The main purpose of this paper is to establish the following global existence and uniqueness of a mild solution to the FNS equations with a small initial data in the critical Besov-Q space.

**Theorem 0.1.** *Given*

$$\begin{cases} \beta > \frac{1}{2}; \\ 1 < p, q < \infty; \\ \gamma_1 = \gamma_2 - 2\beta + 1; \\ m > \max\{p, \frac{n}{2\beta}\}; \\ 0 < m' < \min\{1, \frac{p}{2\beta}\}. \end{cases}$$

*If the index  $(\beta, p, \gamma_2)$  obeys*

$$1 < p \leq 2 \quad \& \quad \frac{2\beta - 2}{p} < \gamma_2 \leq \frac{n}{p}$$

*or*

$$2 < p < \infty \quad \& \quad \beta - 1 < \gamma_2 \leq \frac{n}{p}$$

*then (0.1) has a unique global mild solution in  $(\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2})^n$  for any initial data  $a$  with  $\|a\|_{(\dot{B}_{p,q}^{\gamma_1,\gamma_2})^n}$  being small. Here the symbols  $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ , and  $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$  stand for the so-called Besov-Q spaces and their induced tent spaces, and will be determined properly in Sections 2 and 3.*

Needless to say, our current work grows from the already-known results. In [21] Lions proved the global existence of the classical solutions of (0.1) when  $\beta \geq \frac{5}{4}$  and  $n = 3$ . This existence result was extended to  $\beta \geq \frac{1}{2} + \frac{n}{4}$  by Wu [30], and moreover, for the important case  $\beta < \frac{1}{2} + \frac{n}{4}$ , Wu [31, 32] established the global existence for (0.1) in the Besov spaces  $\dot{B}_p^{1+\frac{n}{p}-2\beta,q}(\mathbb{R}^n)$  for  $1 \leq q \leq \infty$  and for either  $\frac{1}{2} < \beta$  and  $p = 2$  or  $\frac{1}{2} < \beta \leq 1$  and  $2 < p < \infty$  and in  $\dot{B}_2^{r,\infty}(\mathbb{R}^n)$  with  $r > \max\{1, 1 + \frac{n}{p} - 2\beta\}$ ; see also [33] concerning the corresponding regularity. Importantly, Koch-Tataru [15] studied the global existence and uniqueness of (0.1) with  $\beta = 1$  via introducing  $BMO^{-1}(\mathbb{R}^n)$ ; see Miao-Yuan-Zhang [22] for a related account. Generalizing Koch-Tataru's work [15], Xiao [35] introduced the Q-spaces  $Q_\alpha^{-1}(\mathbb{R}^n)$ ,  $0 < \alpha < 1$  to investigate the global existence and uniqueness of the classical Navier-Stokes equations. The ideas of [35] were developed by Li-Zhai [18] to study the

global existence and uniqueness of (0.1) with an initial data being small in  $Q_{\alpha}^{\beta,-1}(\mathbb{R}^n)$  via defining a class of Q-type spaces  $Q_{\alpha}^{\beta}(\mathbb{R}^n)$  under  $\beta \in (\frac{1}{2}, 1)$ . Recently, Lin and Yang [20] got the global existence and uniqueness of (0.1) with initial data being small in some diagonal Besov-Q spaces for  $\beta \in (\frac{1}{2}, 1)$ .

In fact, the above historical citations lead us to make a decisive two-fold observation. On the one hand, thanks to that (0.1) is invariant under the scaling

$$\begin{cases} u_{\lambda}(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x); \\ p_{\lambda}(t, x) = \lambda^{4\beta-2} p(\lambda^{2\beta} t, \lambda x), \end{cases}$$

the initial data space  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  is critical for (0.1) in the sense that the space is invariant under the scaling

$$(0.3) \quad f_{\lambda}(x) = \lambda^{2\beta-1} f(\lambda x).$$

A simple computation, along with letting  $\beta = 1$  in (0.3), indicates that the function spaces:

$$\begin{cases} \dot{L}_{\frac{n}{2}-1}^2(\mathbb{R}^n) = \dot{B}_2^{-1+\frac{n}{2}, 2}(\mathbb{R}^n); \\ L^n(\mathbb{R}^n); \\ \dot{B}_p^{-1+\frac{n}{p}, q}(\mathbb{R}^n); \\ BMO^{-1}(\mathbb{R}^n), \end{cases}$$

are critical for (0.1) with  $\beta = 1$ ; moreover, (0.3) under  $\beta > 1/2$  is valid for functions in the homogeneous Besov spaces  $\dot{B}_2^{1+\frac{n}{2}-2\beta, 1}(\mathbb{R}^n)$  and  $\dot{B}_2^{1+\frac{n}{2}-2\beta, \infty}(\mathbb{R}^n)$  attached to (0.1). On the other hand, it is appropriate to mention that  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  are essentially the same as the Yang-Yuan's Besov-type spaces introduced in [36], and actually larger than the nearly all critical spaces in the known results, e.g.,

$$\begin{cases} \dot{B}_{p,q}^{\gamma_1, \frac{n}{p}} = \dot{B}_p^{\gamma_1, q}(\mathbb{R}^n) \text{ for } 1 \leq p, q < \infty \text{ \& } -\infty < \gamma_1 < \infty; \\ \dot{B}_{p_0, q_0}^{1+\frac{n}{p}-2\beta, \frac{n}{p}} \supseteq \dot{B}_p^{1+\frac{n}{p}-2\beta, q}(\mathbb{R}^n) \text{ for } 1 < p \leq p_0 \text{ \& } 1 < q \leq q_0 < \infty \text{ \& } \beta > 0; \\ \dot{B}_{2,2}^{\alpha-\beta+1, \alpha+\beta-1} = Q_{\alpha}^{\beta}(\mathbb{R}^n) \text{ for } \alpha \in (0, 1) \text{ \& } \beta \in (1/2, 1) \text{ \& } \alpha + \beta - 1 \geq 0. \end{cases}$$

In order to briefly describe the argument for Theorem 0.1, we should point out that the just-mentioned function spaces have a common trait in the structure, i.e., these spaces can be seen as the Q-spaces with  $L^2$  norm, and the advantage of such spaces is that Fourier transform plays an important role in estimating the bilinear term on the corresponding solution spaces. Nevertheless, for the global existence and uniqueness of a mild solution to (0.1) with a small initial data in  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , we have to seek a new approach. Generally speaking, a mild solution of (0.1) is obtained by using the following method. Assume that the initial data belongs to  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}(\mathbb{R}^n)$ . Via the

iteration process:

$$\begin{cases} u^{(0)}(t, x) = e^{-t(-\Delta)^\beta} a(x); \\ u^{(j+1)}(t, x) = u^{(0)}(t, x) - B(u^{(j)}, u^{(j)})(t, x) \quad \text{for } j = 0, 1, 2, \dots, \end{cases}$$

we construct a contraction mapping on a space in  $\mathbb{R}_+^{1+n}$ , denoted by  $X(\mathbb{R}_+^{1+n})$ . With the initial data being small, the fixed point theorem implies that there exists a unique mild solution of (0.1) in  $X(\mathbb{R}_+^{1+n})$ . More precisely, the iteration is based on the following:

- (i) If  $f$  is in  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , then the function  $(t, x) \mapsto e^{-t(-\Delta)^\beta} f(x)$  belongs to  $X(\mathbb{R}_+^{1+n})$ .
- (ii) The bilinear operator

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P} \nabla(u \otimes v) ds$$

is bounded from  $(X(\mathbb{R}_+^{1+n}))^n \times (X(\mathbb{R}_+^{1+n}))^n$  to  $(X(\mathbb{R}_+^{1+n}))^n$ .

For (i), we choose  $X(\mathbb{R}_+^{1+n}) = \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$  associated with the Besov-Q space  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ . Applying multi-resolution analysis, we prove the semigroup  $e^{-t(-\Delta)^\beta}$  is bounded from  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  to  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ . For the converse, we define an operator  $\pi_\phi$  and then prove

$$f(\cdot, \cdot) \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2} \implies \pi_\phi f(\cdot, \cdot) \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}.$$

For (ii), by applying multi-resolution analysis, we obtain an estimate from  $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n \times (\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$  to  $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$ .

*Remark 0.2.*

(i) Our initial spaces in Theorem 0.1 include both Besov spaces and Q-spaces. For example, Li-Zhai [18], Lin-Yang [20], Wu [30, 31, 32, 33] and Xiao [35]. Many classical function spaces are the special cases of Besov-Q spaces. For example,  $\dot{B}_p^{1+\frac{n}{p}-2\beta, q}(\mathbb{R}^n)$  in Wu [30, 31, 32, 33],  $Q_\alpha^\beta(\mathbb{R}^n)$  in Xiao [35] and Li-Zhai [18]. Moreover, in [18, 20], the scope of  $\beta$  is  $(\frac{1}{2}, 1)$ . Our method is valid for  $\beta > \frac{1}{2}$ .

(ii) Moreover, our initial spaces in Theorem 0.1 include a lot of new spaces which are not contained in the above known results. Meanwhile, our initial spaces are larger than the initial spaces introduced in most of the known results. See also Lemma 2.10, Corollary 2.11 and Remark 2.12.

(iii) Furthermore, if  $m$  is chosen to be a big positive real number, then the functions in  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}(\mathbb{R}_+^{1+n})$  are sufficient smooth and have high regularity. Hence the regularity of our solutions in Theorem 0.1 is higher than that of the previous solutions.

The remaining of this paper is organized as follows. In Section 1, we list some preliminary knowledge on wavelets and give the wavelet characterization of the Besov spaces. In Sections 2-3 we define the initial data spaces

and the corresponding solution spaces. Section 4 carries out a necessary analysis of some non-linear terms and a priori estimates. In Section 5, we verify Theorem 0.1 via five lemmas which will be demonstrated in the final Sections 6-7-8-9-10.

*Notations:*  $U \approx V$  represents that there is a constant  $c > 0$  such that  $c^{-1}V \leq U \leq cV$  whose right inequality is also written as  $U \lesssim V$ . Similarly, one writes  $V \gtrsim U$  for  $V \geq cU$ .

## 1. MEYER WAVELETS AND BESOV SPACES

In this section, we recall some basic properties about wavelets and give a wavelet characterization of Besov spaces.

**1.1. Meyer and Daubechies wavelets.** First of all, we indicate that we will use tensorial product real-valued orthogonal wavelets, they may be regular Daubechies wavelets and classical Meyer wavelets. The Daubechies wavelets are used only for the characterization of Besov spaces and Besov-Q spaces, but Meyer spaces will be used almost throughout this paper. Note that the regular Daubechies wavelets are such Daubechies wavelets that are smooth enough and have more sufficient vanishing moments than the relative spaces do. See Lemma 1.5 and the part before Lemma 2.2.

Then we present some preliminaries on Meyer wavelets  $\Phi^\epsilon(x)$  in detail and refer the reader to [23], [29] and [37] for further information. Let  $\Psi^0$  be an even function in  $C_0^\infty([-\frac{4\pi}{3}, \frac{4\pi}{3}])$  with

$$\begin{cases} 0 \leq \Psi^0(\xi) \leq 1; \\ \Psi^0(\xi) = 1 \text{ for } |\xi| \leq \frac{2\pi}{3}. \end{cases}$$

From now on, let

$$\Omega(\xi) = \sqrt{(\Psi^0(\frac{\xi}{2}))^2 - (\Psi^0(\xi))^2}.$$

Then  $\Omega(\xi)$  is an even function in  $C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$ . Clearly,

$$\begin{cases} \Omega(\xi) = 0 \text{ for } |\xi| \leq \frac{2\pi}{3}; \\ \Omega^2(\xi) + \Omega^2(2\xi) = 1 = \Omega^2(\xi) + \Omega^2(2\pi - \xi) \text{ for } \xi \in [\frac{2\pi}{3}, \frac{4\pi}{3}]. \end{cases}$$

Let  $\Psi^1(\xi) = \Omega(\xi)e^{-\frac{i\xi}{2}}$ . For any  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ , define  $\Phi^\epsilon(x)$  by  $\hat{\Phi}^\epsilon(\xi) = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i)$ . For  $j \in \mathbb{Z}$  and  $k \in \mathbb{Z}^n$ , set  $\Phi_{j,k}^\epsilon(x) = 2^{\frac{nj}{2}} \Phi^\epsilon(2^j x - k)$ .

Furthermore, we put

$$\begin{cases} E_n = \{0, 1\}^n \setminus \{0\}; \\ F_n = \{(\epsilon, k) : \epsilon \in E_n, k \in \mathbb{Z}^n\}; \\ \Lambda_n = \{(\epsilon, j, k), \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \end{cases}$$

and for any  $\epsilon \in \{0, 1\}^n, k \in \mathbb{Z}^n$  and a function  $f$  on  $\mathbb{R}^n$ , we write  $f_{j,k}^\epsilon = \langle f, \Phi_{j,k}^\epsilon \rangle$ .

The following result is well-known.

**Lemma 1.1.** *The Meyer wavelets  $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$  form an orthogonal basis in  $L^2(\mathbb{R}^n)$ . Consequently, for any  $f \in L^2(\mathbb{R}^n)$ , the following wavelet decomposition holds in the  $L^2$  convergence sense:*

$$f = \sum_{(\epsilon,j,k) \in \Lambda_n} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

Moreover, for  $j \in \mathbb{Z}$ , let

$$P_j f = \sum_{k \in \mathbb{Z}^n} f_{j,k}^0 \Phi_{j,k}^0 \quad \text{and} \quad Q_j f = \sum_{(\epsilon,k) \in F_n} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

For the above Meyer wavelets, the product of any two functions  $u$  and  $v$  can be decomposed as

$$(1.1) \quad \begin{aligned} uv = & \sum_{j \in \mathbb{Z}} P_{j-3} u Q_j v + \sum_{j \in \mathbb{Z}} Q_j u Q_j v + \sum_{0 < j-j' \leq 3} Q_j u Q_{j'} v \\ & + \sum_{0 < j'-j \leq 3} Q_j u Q_{j'} v + \sum_{j \in \mathbb{Z}} Q_j u P_{j-3} v. \end{aligned}$$

**1.2. Wavelet characterization of Besov spaces.** Suppose that  $\varphi$  is a function on  $\mathbb{R}^n$  such that

$$\begin{cases} \text{supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}; \\ \hat{\varphi}(\xi) = 1 \text{ for } \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{2}\}. \end{cases}$$

Define

$$\varphi_v(x) = 2^{n(v+1)} \varphi(2^{v+1}x) - 2^{nv} \varphi(2^v x) \quad \forall v \in \mathbb{Z},$$

as the Littlewood-Paley functions. The set of functions  $\{\varphi_v\}_{v \in \mathbb{Z}}$  enjoys

$$\begin{cases} \varphi_v(x) \in \mathbb{R}^n; \\ \text{supp } \hat{\varphi}_v \subset \{\xi \in \mathbb{R}^n, \frac{1}{2} \leq 2^{-v} |\xi| \leq 2\}; \\ |\hat{\varphi}_v(\xi)| \geq C > 0 \text{ for } \frac{3}{5} \leq 2^{-v} |\xi| \leq \frac{5}{3}; \\ |\partial^\alpha \hat{\varphi}_v(\xi)| \leq C_\alpha 2^{-v|\alpha|} \text{ for } \alpha \geq 0; \\ \sum_{v=-\infty}^{\infty} \hat{\varphi}_v(\xi) = 1. \end{cases}$$

**Definition 1.2.** Given  $-\infty < \alpha < \infty$ ,  $0 < p, q < \infty$ . A function  $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$  belongs to  $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$  if

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \|\varphi_v * f\|_p^q \right]^{\frac{1}{q}} < \infty.$$

If  $q = \infty$ , the Besov spaces  $\dot{B}_p^{\alpha,\infty}(\mathbb{R}^n)$  is defined as follows.

**Definition 1.3.** Given  $-\infty < \alpha < \infty$  and  $0 < p \leq \infty$ . A function  $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$  belongs to  $\dot{B}_p^{\alpha,\infty}(\mathbb{R}^n)$  if

$$\|f\|_{\dot{B}_p^{\alpha,\infty}} = \sup_{v \in \mathbb{Z}} 2^{v\alpha} \|\varphi_v * f\|_p < \infty.$$

According to Peetre's paper [26], this definition of  $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$  is independent of the choice of  $\{\varphi_v\}_{v \in \mathbb{Z}}$ . Let  $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$  be Meyer wavelets defined above. We can characterize  $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$  as follows.

**Theorem 1.4.** Given  $s \in \mathbb{R}$  and  $0 < p, q < \infty$ . A function  $f$  belongs to  $\dot{B}_p^{s,q}(\mathbb{R}^n)$  if and only if

$$\left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon,k} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty.$$

*Proof.* Given the Littlewood-Paley functions  $\{\varphi_v\}_{v \in \mathbb{Z}^n}$  and Meyer wavelets  $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$ . Then for  $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ , we can get  $\{f_v\}_{v \in \mathbb{Z}}$  and  $\{f_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$ . Hence

$$\begin{aligned} I &= \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \|f_v\|_p^q \right]^{\frac{1}{q}} \\ &= \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \|f * \varphi_v\|_p^q \right]^{\frac{1}{q}} \\ &\leq \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \left\| \sum_{(\epsilon,j,k) \in \Lambda_n} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon * \varphi_v \right\|_p^q \right]^{\frac{1}{q}}. \end{aligned}$$

Since  $\widehat{\Phi}^\epsilon$  and  $\widehat{\varphi}$  are compactly supported, there exists a constant  $c > 0$  such that

$$|j - v| > c \implies \Phi_{j,k}^\epsilon * \varphi_v(x) = 0.$$

Then we have

$$\begin{aligned} I &= \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \left\| \sum_{|j-v| \leq c} \sum_{\epsilon,k} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon * \varphi_v \right\|_p^q \right]^{\frac{1}{q}} \\ &\leq \left[ \sum_{v \in \mathbb{Z}} 2^{qv\alpha} \left( \sum_{|j-v| \leq c} \left\| \sum_{\epsilon,k} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon * \varphi_v \right\|_p \right)^q \right]^{\frac{1}{q}}. \end{aligned}$$

Because

$$\left\| \sum_{\epsilon,k} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon * \varphi_v \right\|_p \lesssim \left\| \sum_{\epsilon,k} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon \right\|_p \lesssim \left( \sum_{\epsilon,k} 2^{nj(\frac{n}{2}-1)} |f_{j,k}^\epsilon|^p \right)^{\frac{1}{p}},$$



we obtain

$$\begin{aligned}
I &\lesssim \left[ \sum_{v \in \mathbb{Z}} 2^{qvs} \left( \sum_{|j-v| \leq c} \sum_{\epsilon, k} 2^{nj(\frac{p}{2}-1)} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\lesssim \left[ \sum_{v \in \mathbb{Z}} \sum_{|j-v| \leq c} 2^{qvs} \left( \sum_{\epsilon, k} 2^{nj(\frac{p}{2}-1)} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\lesssim \left[ \sum_{v \in \mathbb{Z}} \sum_{|j-v| \leq c} 2^{qjs} \left( \sum_{\epsilon, k} 2^{nj(\frac{p}{2}-1)} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\lesssim \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
\end{aligned}$$

Conversely, we have

$$\begin{aligned}
J &= \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |f_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |\langle f, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |\langle \sum_{v \in \mathbb{Z}} f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |\langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
\end{aligned}$$

Because the supports of  $\widehat{\phi_{j,k}^\epsilon}$  and  $\widehat{f * \varphi_v}$  are compact, by the Parseval identity, there exists a constant  $c > 0$  such that

$$|v - j| > c \implies \langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle = 0.$$

Hence we get

$$\begin{aligned}
J &= \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |\langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\lesssim \left[ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{|v-j| \leq c} \sum_{\epsilon, k} |\langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\lesssim \left[ \sum_{j \in \mathbb{Z}} \sum_{|v-j| \leq C} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} |\langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
\end{aligned}$$

Because

$$\left( \sum_{\epsilon, k} |\langle f * \varphi_v, \Phi_{j,k}^\epsilon \rangle|^p \right)^{\frac{1}{p}} \lesssim 2^{nj(\frac{1}{p}-\frac{1}{2})} \|f * \varphi_v\|_p,$$

we obtain

$$J \lesssim \left( \sum_{j \in \mathbb{Z}} \sum_{|j-v| \leq c} 2^{qjs} \|f * \varphi_v\|_p^q \right)^{\frac{1}{q}} \lesssim \left[ \sum_{v \in \mathbb{Z}} 2^{qvs} \|f_v\|_p^q \right]^{\frac{1}{q}}.$$

□

Now we prove that the above wavelet characterization is independent on the choice of wavelet functions. For this purpose, we need a preliminary

lemma. Let  $\{\Phi_{j,k}^{\epsilon,1}\}_{(\epsilon,j,k) \in \Lambda_n}$  and  $\{\Phi_{j,k}^{\epsilon,2}\}_{(\epsilon,j,k) \in \Lambda_n}$  be two different wavelet bases. Denote

$$a_{j,k,j',k'}^{\epsilon,\epsilon'} = \langle \Phi_{j,k}^{\epsilon,1}, \Phi_{j',k'}^{\epsilon',2} \rangle.$$

We have the following lemma.

**Lemma 1.5.** *Let  $\{\Phi_{j,k}^{\epsilon,1}\}_{(\epsilon,j,k) \in \Lambda_n}$  and  $\{\Phi_{j,k}^{\epsilon,2}\}_{(\epsilon,j,k) \in \Lambda_n}$  be two different wavelet bases which are sufficiently regular. Then for any natural number  $N$  there exists a positive constant  $C_N$  such that for  $j, j' \in \mathbb{Z}$  and  $k, k' \in \mathbb{Z}^n$ ,*

$$(1.2) \quad |a_{j,k,j',k'}^{\epsilon,\epsilon'}| \leq C_N 2^{-|j-j'|(\frac{n}{2}+N)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N}.$$

*Proof.* We only consider the case  $j \geq j'$ . The proof for  $j < j'$  is similar. For  $\epsilon \neq 0$ , denote by  $i_\epsilon$  the smallest one of the indexes  $i$  with  $\epsilon_i \neq 0$ . For  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , write  $y_0 = x_{i_\epsilon}$ . For any  $N \geq 1$ , denote by  $x_N^\epsilon$  the vector  $(x_1, \dots, x_{i_\epsilon-1}, y_N, x_{i_\epsilon+1}, \dots, x_n)$ . For any  $\Phi$ , write  $I_N^\epsilon \Phi(x) = \int_{-\infty}^{x_{i_\epsilon}} \dots \int_{-\infty}^{y_{N-1}} \Phi(x_N^\epsilon) dy_1 \dots dy_N$ . Denote by  $D_N^\epsilon \Phi$  the function  $\frac{\partial^N}{\partial x_{i_\epsilon}^N} \Phi$ . We can get

$$\begin{aligned} a_{j,k,j',k'}^{\epsilon,\epsilon'} &= 2^{\frac{n(j'-j)}{2}} \langle \Phi^{\epsilon,1}, \Phi^{\epsilon',2}(2^{j'-j} \cdot -k' + 2^{j'-j}k) \rangle \\ &= 2^{(\gamma+\frac{n}{2})(j'-j)} \langle I_\gamma^\epsilon \Phi^{\epsilon,1}, (D_\gamma^{\epsilon'} \Phi^{\epsilon',2})(2^{j'-j} \cdot -k' + 2^{j'-j}k) \rangle \\ &=: b_{j-j',k-2^{j-j'}k'}^{\epsilon,\epsilon'}. \end{aligned}$$

By the smoothness and canceling properties of wavelet function,  $I_N^\epsilon \Phi$  and  $D_N^\epsilon \Phi$  still decay very fast. Hence we can obtain

$$\begin{aligned} |a_{j,k,j',k'}^{\epsilon,\epsilon'}| &\lesssim \int_{\mathbb{R}^n} (1+|x|)^{-N_1} (1+|2^{j'-j}x - k' + 2^{j'-j}k|)^{-N_2} dx \\ &\lesssim \int_{|x| \leq 2^{j-j'-1}|k'-2^{j'-j}k|} (1+|x|)^{-N_1} (1+|k' - 2^{j'-j}k|)^{-N_2} dx \\ &\quad + \int_{|x| > 2^{j-j'-1}|k'-2^{j'-j}k|} (1+|x|)^{-N_1} dx. \end{aligned}$$

Taking  $N_1 > N + n$  and  $N_2 > N + n$ , we can get the desired estimate. This completes the proof of Lemma 1.5.  $\square$

By Lemma 1.5, we get

**Theorem 1.6.** *Given  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . The wavelet characterization of  $\dot{B}_p^{s,q}(\mathbb{R}^n)$  is independent to the choice of wavelet functions.*

*Proof.* We consider only the proof for  $0 < p, q < \infty$ . For the cases where  $p = \infty$  or  $q = \infty$ , we make some suitable modification.

In what follows, set

$$u_j = \left( \sum_{\epsilon,k} |a_{j,k}^\epsilon|^p \right)^{\frac{1}{p}}.$$

To prove this theorem, we only need to prove the following estimate:

$$(1.3) \quad I = \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon, k} \left| \sum_{\epsilon', j', k'} a_{j, k, j', k'}^{\epsilon, \epsilon'} a_{j', k'}^{\epsilon'} \right|^p \right)^{\frac{q}{p}} \lesssim \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q.$$

Now we divide the rest of the proof into two cases.

Case 1:  $0 < p \leq 1$ . For this case,

$$\begin{aligned} I &\leq \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{\epsilon', j', k'} \sum_{\epsilon, k} |a_{j, k, j', k'}^{\epsilon, \epsilon'}|^p |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\ &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' \leq j} \sum_{\epsilon', k'} \sum_{\epsilon, k} |a_{j, k, j', k'}^{\epsilon, \epsilon'}|^p |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\ &\quad + \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' > j} \sum_{\epsilon', k'} \sum_{\epsilon, k} |a_{j, k, j', k'}^{\epsilon, \epsilon'}|^p |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{q}{p}}. \end{aligned}$$

By (1.2), we have

$$\begin{aligned} I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' \leq j} 2^{-(p\gamma+\frac{pm}{2}-n)(j-j')} u_{j'}^p \right)^{\frac{q}{p}} \\ &\quad + \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' > j} 2^{-p(\gamma+\frac{n}{2})|j-j'|} u_{j'}^p \right)^{\frac{q}{p}}. \end{aligned}$$

If  $q \leq p$ , we have

$$\begin{aligned} I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' \leq j} 2^{-\frac{q}{p}(p\gamma+\frac{pm}{2}-n)(j-j')} u_{j'}^q \\ &\quad + \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' > j} 2^{-q(\gamma+\frac{n}{2})|j-j'|} u_{j'}^q \\ &\lesssim \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' \leq j} 2^{q(s-\gamma)(j-j')} \\ &\quad + \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' > j} 2^{-q(s+\gamma+n-\frac{n}{p})|j-j'|}. \end{aligned}$$

Notice that  $(\frac{1}{p} - 1)n - \gamma < s < \gamma$ . So we can get (1.3).

If  $q > p$ , via taking  $\delta > 0$  small enough we have

$$\begin{aligned} I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' \leq j} 2^{[\delta-\frac{q}{p}(p\gamma+\frac{pm}{2}-n)](j-j')} u_{j'}^q \\ &\quad + \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' > j} 2^{[\delta-q(\gamma+\frac{n}{2})](j-j')} u_{j'}^q \\ &\lesssim \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' \leq j} 2^{[q(s-\gamma)-\delta](j-j')} \\ &\quad + \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' > j} 2^{[\delta-q(s+\gamma+n-\frac{n}{p})](j-j')}. \end{aligned}$$

Because  $(\frac{1}{p} - 1)n - \gamma < s < \gamma$  and  $\delta$  is a sufficiently small positive constant, we can get (1.3).

Case 2:  $p > 1$ . Write  $p' = \frac{p}{p-1}$ . Then  $0 < \tilde{p} =: \frac{p-1}{p} < 1$ . By Hölder's inequality, we can get

$$\begin{aligned} I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \left[ \sum_{j' \geq j} \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{\tilde{p}p'} \right)^{\frac{1}{\tilde{p}'}} \right. \right. \\ &\quad \left. \left. \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{1}{\tilde{p}}} \right]^p \right\}^{\frac{q}{p}} \\ &+ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \left[ \sum_{j' < j} \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{\tilde{p}p'} \right)^{\frac{1}{\tilde{p}'}} \right. \right. \\ &\quad \left. \left. \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{1}{\tilde{p}}} \right]^p \right\}^{\frac{q}{p}}. \end{aligned}$$

By (1.2), we have

$$\begin{aligned} |a_{j, j', k, k'}^{\epsilon, \epsilon'}| &\lesssim 2^{-|j-j'|(\frac{n}{2}+N)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N} \\ &\approx 2^{-|j-j'|(\frac{n}{2}+N)} \left( \frac{2^{j'-j} + 1}{2^{j'-j} + 1 + |2^{j'-j}k - k'|} \right)^{n+N}. \end{aligned}$$

If  $j' < j$ , then  $2^{j'-j} < 1$  and

$$|a_{j, j', k, k'}^{\epsilon, \epsilon'}| \lesssim 2^{-(j-j')(\frac{n}{2}+N)} \left( \frac{2}{1+|2^{j'-j}k - k'|} \right)^{n+N}.$$

If  $j' \geq j$ , then  $2^{j'-j} \geq 1$  and

$$\begin{aligned} |a_{j, j', k, k'}^{\epsilon, \epsilon'}| &\lesssim 2^{-|j-j'|(\frac{n}{2}+N)} \left( \frac{2^{j'-j} + 1}{2^{j'-j} + 1 + |2^{j'-j}k - k'|} \right)^{n+N} \\ &\lesssim 2^{(j'-j)(\frac{n}{2}+N)} \left( \frac{2^{j'-j} + 1}{2^{j'-j} + 1 + |2^{j'-j}k - k'|} \right)^n \\ &\lesssim 2^{(j'-j)(\frac{n}{2}+N)} 2^{n(j'-j)} \left( \frac{1}{1 + |2^{j'-j}k - k'|} \right)^n. \end{aligned}$$

The above estimates for  $|a_{j, j', k, k'}^{\epsilon, \epsilon'}|$  imply that

$$\begin{aligned} I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \left[ \sum_{j' \geq j} 2^{[\frac{n(p-1)}{p}-\tilde{p}(\frac{n}{2}+\gamma)](j'-j)} \right. \right. \\ &\quad \left. \left. \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{1}{\tilde{p}}} \right]^p \right\}^{\frac{q}{p}} \\ &+ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \left[ \sum_{j' < j} 2^{[\tilde{p}(\frac{n}{2}+\gamma)](j'-j)} \right. \right. \\ &\quad \left. \left. \left( \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right)^{\frac{1}{\tilde{p}}} \right]^p \right\}^{\frac{q}{p}}. \end{aligned}$$

Upon letting  $\delta > 0$  be small enough, we use Hölder's inequality to derive

$$\begin{aligned}
I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \sum_{j' \geq j} 2^{[\frac{n(p-1)}{p}-\tilde{p}(\frac{n}{2}+\gamma)+\delta](j'-j)p} \right. \\
&\quad \left. \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right\}^{\frac{q}{p}} \\
&+ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{\epsilon, k} \sum_{j' < j} 2^{[\tilde{p}(\frac{n}{2}+\gamma)-\delta](j'-j)p} \right. \\
&\quad \left. \sum_{\epsilon', k'} |a_{j, j', k, k'}^{\epsilon, \epsilon'}|^{(1-\tilde{p})p} |a_{j', k'}^{\epsilon'}|^p \right\}^{\frac{q}{p}} \\
&\lesssim \sum_{j \in \mathbb{Z}^n} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' \geq j} 2^{-[p(\frac{n}{2}+\gamma)-(p-1)n-\delta](j'-j)} u_{j'}^p \right)^{\frac{q}{p}} \\
&+ \sum_{j \in \mathbb{Z}^n} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \left( \sum_{j' < j} 2^{[p(\frac{n}{2}+\gamma)-n-\delta](j'-j)} u_{j'}^p \right)^{\frac{q}{p}}.
\end{aligned}$$

If  $q \leq p$ , then

$$\begin{aligned}
I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' \geq j} 2^{-\frac{q}{p}[p(\frac{n}{2}+\gamma)-(p-1)n-\delta](j'-j)} u_{j'}^q \\
&+ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' < j} 2^{\frac{q}{p}[p(\frac{n}{2}+\gamma)-n-\delta](j'-j)} u_{j'}^q \\
&\lesssim \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' \geq j} 2^{q(s-\gamma+n-\frac{\delta}{p})(j'-j)} \\
&+ \sum_{j' \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' < j} 2^{-q(s+\frac{\delta}{p}-\gamma)(j'-j)}
\end{aligned}$$

which implies (1.3) by the facts that  $s < \gamma - n$  and  $\delta$  is small enough.

If  $q > p$ , then taking a positive constant  $\delta'$  small enough we have

$$\begin{aligned}
I &\lesssim \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' \geq j} 2^{-\frac{q}{p}[p(\frac{n}{2}+\gamma)-(p-1)n-\delta-\delta'](j'-j)} u_{j'}^q \\
&+ \sum_{j \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} \sum_{j' < j} 2^{\frac{q}{p}[p(\frac{n}{2}+\gamma)-n-\delta-\delta'](j'-j)} u_{j'}^q \\
&\lesssim \sum_{j' \in \mathbb{Z}} 2^{qj'(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' \geq j} 2^{q(s-\gamma+n-\frac{\delta+\delta'}{p})(j'-j)} \\
&+ \sum_{j' \in \mathbb{Z}} 2^{qj(s+\frac{n}{2}-\frac{n}{p})} u_{j'}^q \sum_{j' < j} 2^{-q(s+\frac{\delta+\delta'}{p}-\gamma)(j'-j)},
\end{aligned}$$

whence reaching (1.3), and therefore completing the proof of the theorem.  $\square$

## 2. BESOV-Q SPACES VIA WAVELETS

**2.1. Definition and its wavelet formulation.** As a generalization of the Morrey spaces, the forthcoming Besov-Q spaces cover many important function spaces, for example, Besov spaces, Morrey spaces and Q-spaces and so on. Such spaces were first introduced by wavelets in Yang [37]. On the other hand, our Lemma 2.2 as below and Yang-Yuan's [36, Theorem 3.1] show that our Besov-Q spaces and their Besov type spaces coincide;

see also Liang-Sawano-Ullrich-Yang-Yuan [19] and Yuan-Sickel-Yang [39] for more information on the so-called Yang-Yuan's spaces.

Let  $\varphi \in C_0^\infty(B(0, n))$  and  $\varphi(x) = 1$  for  $x \in B(0, \sqrt{n})$ . Let  $Q(x_0, r)$  be a cube parallel to the coordinate axis, centered at  $x_0$  and with side length  $r$ . For simplicity, sometimes, we denote by  $Q = Q(r)$  the cube  $Q(x_0, r)$  and let  $\varphi_Q(x) = \varphi(\frac{x-x_Q}{r})$ . For  $1 < p, q < \infty$  and  $\gamma_1, \gamma_2 \in \mathbb{R}$ , let  $m_0 = m_{p,q}^{\gamma_1, \gamma_2}$  be a positive constant large enough. For arbitrary function  $f$ , let  $S_{p,q,f}^{\gamma_1, \gamma_2}$  be the class of the polynomial functions  $P_{Q,f}$  such that

$$\int x^\alpha \varphi_Q(x)(f(x) - P_{Q,f}(x))dx = 0 \quad \forall \quad |\alpha| \leq m_0.$$

**Definition 2.1.** Given  $1 < p, q < \infty$  and  $\gamma_1, \gamma_2 \in \mathbb{R}$ . We say that  $f$  belongs to the Besov-Q space  $\dot{B}_{p,q}^{\gamma_1, \gamma_2} := \dot{B}_{p,q}^{\gamma_1, \gamma_2}(\mathbb{R}^n)$  provided

$$(2.1) \quad \sup_Q |Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \inf_{P_{Q,f} \in S_{p,q,f}^{\gamma_1, \gamma_2}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{B}_p^{\gamma_1, q}} < \infty,$$

where the supremum is taken over all cubes  $Q$  with center  $x_Q$  and length  $r$ .

Given  $1 < p, q < \infty$  and  $\gamma_1, \gamma_2 \in \mathbb{R}$ . Let  $m_0 = m_{p,q}^{\gamma_1, \gamma_2}$  be a sufficiently big integer. For the regular Daubechies wavelets  $\Phi^\epsilon(x)$ , there exist two integers  $m \geq m_0 = m_{p,q}^{\gamma_1, \gamma_2}$  and  $M$  such that for  $\epsilon \in E_n$ ,  $\Phi^\epsilon(x) \in C_0^m([-2^M, 2^M]^n)$  and  $\int x^\alpha \Phi^\epsilon(x)dx = 0 \quad \forall \quad |\alpha| \leq m$ . By applying the regular Daubechies wavelets, we have the following wavelet characterization for  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ .

**Lemma 2.2.**  $f = \sum_{\epsilon, j, k} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in \dot{B}_{p,q}^{\gamma_1, \gamma_2} \iff$

$$(2.2) \quad \sup_Q |Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \left[ \sum_{n_j \geq -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < +\infty,$$

where the supremum is taken over all dyadic cubes in  $\mathbb{R}^n$ .

*Proof.* At first we assume that  $f \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}(\mathbb{R}^n)$ . For any dyadic cube  $Q$  with center  $x_Q$  and side length  $l(Q)$ , there exists a cube  $\tilde{Q}$ , parallel to the coordinate axis, centered at  $x_Q$  and with side length  $2^{M+2}l(Q)$ . For such  $\tilde{Q}$ , for any  $\epsilon \in E_n$ ,  $Q_{j,k} \subset Q$ ,  $x \in \tilde{Q}$ , by definition of  $\phi_Q(x)$  and the vanishing moments of Daubechies wavelets, we have  $f(x) = \phi_{\tilde{Q}}(x)f(x)$  and

$$\int (\phi_{\tilde{Q}}(y)P_{\tilde{Q},f}(y))\Phi_{j,k}^\epsilon(y)dy = \int P_{\tilde{Q},f}(y)\Phi_{j,k}^\epsilon(y)dy = 0.$$

Hence, for any  $\epsilon \in E_n$ ,  $Q_{j,k} \subset Q$ , we have

$$\langle f, \Phi_{j,k}^\epsilon \rangle = \langle \phi_{\tilde{Q}}(f - P_{\tilde{Q},f}), \Phi_{j,k}^\epsilon \rangle.$$

That is to say, by the wavelet characterization of Besov spaces, we have

$$\left[ \sum_{n, j \geq -\log_2 |Q|} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\ \lesssim \inf_{P_{Q,f} \in S_{p,q,f}^{\gamma_1, \gamma_2}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{B}_p^{\gamma_1, q}}.$$

Taking the supremum over dyadic cubes on both sides of the above inequality, we can see that (2.2) holds.

Conversely, for any cube  $Q$ , there exists  $2^n$  dyadic cube  $Q_i$  with side length less than  $2^{M+3}l(Q)$  and greater than  $2^{M+2}l(Q)$ , such that

$$\phi_Q f = \phi_Q \sum_{i=1}^{2^n} \sum_{\epsilon \in E_n, Q_{j,k} \subset Q_i} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

That is to say,

$$\|\phi_Q f\|_{\dot{B}_p^{\gamma_1, q}} \leq \left\| \phi_Q \sum_{i=1}^{2^n} \sum_{\epsilon \in E_n, Q_{j,k} \subset Q_i} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon \right\|_{\dot{B}_p^{\gamma_1, q}} \\ \lesssim \left\| \sum_{i=1}^{2^n} \sum_{\epsilon \in E_n, Q_{j,k} \subset Q_i} f_{j,k}^\epsilon \Phi_{j,k}^\epsilon \right\|_{\dot{B}_p^{\gamma_1, q}}.$$

This derives that if (2.2) is true then (2.1) is valid.  $\square$

A direct application of Lemma 2.2 gives the following assertion.

**Corollary 2.3.** *Given  $1 < p, q < \infty, \gamma_1, \gamma_2 \in \mathbb{R}$ .*

- (i) *Each  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  is a Banach space.*
- (ii) *The definition of  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  is independent on the choice of  $\phi$ .*

Now we recall some preliminaries on the Calderón-Zygmund operators (cf. [23] and [24]). For  $x \neq y$ , let  $K(x, y)$  be a smooth function such that there exists a sufficiently large  $N_0 \leq m$  satisfying that

$$(2.3) \quad |\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |x - y|^{-(n+|\alpha|+|\beta|)} \quad \forall \quad |\alpha| + |\beta| \leq N_0.$$

A linear operator

$$Tf(x) = \int K(x, y)f(y)dy$$

is said to be a Calderón-Zygmund one if it is continuous from  $C^1(\mathbb{R}^n)$  to  $(C^1(\mathbb{R}^n))'$ , where the kernel  $K(\cdot, \cdot)$  satisfies (2.3) and

$$Tx^\alpha = T^*x^\alpha = 0 \quad \forall \quad \alpha \in \mathbb{N}^n \quad \text{with} \quad |\alpha| \leq N_0.$$

For such an operator, we denote  $T \in CZO(N_0)$ .

The kernel  $K(\cdot, \cdot)$  may have high singularity on the diagonal  $x = y$ , so according to the Schwartz kernel theorem, it is only a distribution in  $S'(\mathbb{R}^{2n})$ . For  $(\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n$ , let

$$a_{j,k,j',k'}^{\epsilon, \epsilon'} = \langle K(x, y), \Phi_{j,k}^\epsilon(x) \Phi_{j',k'}^{\epsilon'}(y) \rangle.$$

If  $T$  is a Calderón-Zygmund operator, then its kernel  $K(\cdot, \cdot)$  and the related coefficients satisfy the following relations (cf. [23], [24] and [37]):

**Lemma 2.4.**

(i) If  $T \in CZO(N_0)$ , then the coefficients  $a_{j,k,j',k'}^{\epsilon,\epsilon'}$  satisfy

$$(2.4) \quad |a_{j,k,j',k'}^{\epsilon,\epsilon'}| \lesssim \frac{\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+N_0}}{2^{|j-j'|(\frac{n}{2}+N_0)}} \quad \forall (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n.$$

(ii) If  $a_{j,k,j',k'}^{\epsilon,\epsilon'}$  satisfy (2.4), then  $K(\cdot, \cdot)$ , the kernel of the operator  $T$ , can be written as

$$K(x, y) = \sum_{(\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} \Phi_{j,k}^{\epsilon}(x) \Phi_{j',k'}^{\epsilon'}(y)$$

in the distribution sense. Moreover,  $T$  belongs to  $CZO(N_0 - \delta)$  for any small positive number  $\delta$ .

The following tells that the Calderón-Zygmund operators are bounded on the Besov-Q spaces.

**Theorem 2.5.** For  $1 < p, q < \infty, \gamma_1, \gamma_2 \in \mathbb{R}$ , there exists sufficient big  $N_0$  such that  $\{a_{j,k,j',k'}^{\epsilon,\epsilon'}\}_{(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n}$  satisfies (2.4). If  $\{g_{j,k}^{\epsilon}\}_{(\epsilon,j,k) \in \Lambda_n} \subset \dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , then

$$\{\tilde{g}_{j,k}^{\epsilon} \equiv \sum_{(\epsilon',j',k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}\}_{(\epsilon,j,k) \in \Lambda_n} \subset \dot{B}_{p,q}^{\gamma_1, \gamma_2}.$$

*Proof.* According to Lemmas 2.2 & 2.4, we need to prove that  $\{\tilde{g}_{j,k}^{\epsilon}\}_{(\epsilon,j,k) \in \Lambda_n}$  satisfies the inequality (2.2). For similarity, we only prove the case  $p = q$ . For any dyadic cube  $Q$ , we have

$$\begin{aligned} I_g^Q &\equiv |Q|^{\frac{p\gamma_2}{n}-1} \sum_{n_j \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon,k): Q_{j,k} \subset Q} |a_{j,k}^{\epsilon}|^p \\ &\lesssim \|\{g_{j,k}^{\epsilon}\}_{(\epsilon,j,k) \in \Lambda_n}\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}}^p. \end{aligned}$$

Let  $Q$  be any dyadic cube with  $|Q| = 2^{-nj_0}$ . For  $\tau \geq 1$ , denote by  $Q_{\tau}$  the dyadic cube satisfying  $Q \subset Q_{\tau}$  and  $|Q_{\tau}| = 2^{n\tau}|Q|$ . For convenience of the notations, we write also  $Q_0 = Q$ . If  $l \in \mathbb{Z}^n$  and  $Q_{j',k'} \subset 2^{-j_0}l + Q$ , we denote  $Q_{j',k'} \in S_{0,l}$ . If  $\tau \geq 1$ ,  $l \in \mathbb{Z}^n$  and  $Q_{j',k'} \subset 2^{\tau-j_0}l + Q_{\tau}$ , we denote  $Q_{j',k'} \in S_{\tau,l}$ . Then we have

$$\begin{aligned} \sum_{(\epsilon',j',k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'} &= \sum_{\tau \geq 0} \sum_{l \in \mathbb{Z}^n} \sum_{Q_{j',k'} \in S_{\tau,l}} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'} \\ &\equiv \sum_{\tau \geq 0} \sum_{l \in \mathbb{Z}^n} I_{\tau,l}^Q. \end{aligned}$$



Hence for any sufficient small  $\delta > 0$ , we have

$$\begin{aligned} I_g^Q &= |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon, k): Q_{j,k} \subset Q} \left| \sum_{\tau \geq 0} \sum_{l \in \mathbb{Z}^n} I_{\tau, l}^Q \right|^p \\ &\lesssim \sum_{\tau \geq 0} \sum_{l \in \mathbb{Z}^n} 2^{\tau\delta} (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon, k): Q_{j,k} \subset Q} |I_{\tau, l}^Q|^p \\ &\equiv \sum_{\tau \geq 0} \sum_{l \in \mathbb{Z}^n} J_{\tau, l}^Q. \end{aligned}$$

We first consider the case  $\tau = 0$  and distinguish two cases:

Case 1:  $j \geq j'$ . For such case, the condition (2.4) becomes

$$|a_{j,k,j',k'}^{\epsilon, \epsilon'}| \lesssim 2^{-|j-j'|(\frac{n}{2} + N_0)} (1 + |k2^{j'-j} - k'|)^{-(n+N_0)}.$$

Further, for  $l$ , we distinguish two subcases:

Subcase 1.1:  $|l| \leq 8^n$ . For each  $l$ , by the definition of  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , we can see that

$$\left\| \sum_{Q_{j',k'} \subset S_{0,l}} g_{j',k'}^{\epsilon'} \phi_{j',k'}^{\epsilon'} \right\|_{\dot{B}_p^{\gamma_1, q}} \lesssim |Q|^{\frac{1}{p} - \frac{\gamma_2}{n}}.$$

Because Calderón-Zygmund operator is bounded on  $\dot{B}_p^{\gamma_1, q}$ , we can get

$$\left\| \sum_{(\epsilon, k) \in \Lambda_n} \sum_{Q_{j',k'} \subset S_{0,l}} a_{j,k,j',k'}^{\epsilon, \epsilon'} g_{j',k'}^{\epsilon'} \phi_{j',k'}^{\epsilon'} \right\|_{\dot{B}_p^{\gamma_1, q}} \lesssim |Q|^{\frac{1}{p} - \frac{\gamma_2}{n}}.$$

Subcase 1.2:  $|l| > 8^n$ . For this case, if  $Q_{j',k'} \in S_{0,l}$ , we have  $|k2^{j'-j} - k'| \sim 2^{j'-j_0}|l|$  and

$$|a_{j,k,j',k'}^{\epsilon, \epsilon'}| \lesssim 2^{-|j-j'|(\frac{n}{2} + N_0)} (2^{j'-j_0}|l|)^{-(n+N_0)}.$$

Hence we get

$$\begin{aligned} J_{0,l}^Q &\lesssim (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon, k): Q_{j,k} \subset Q} \left| \sum_{Q_{j',k'} \in S_{0,l}} a_{j,k,j',k'}^{\epsilon, \epsilon'} g_{j',k'}^{\epsilon'} \phi_{j',k'}^{\epsilon'} \right|^p \\ &\lesssim (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon, k): Q_{j,k} \subset Q} \left| \sum_{j_0 \leq j' \leq j} 2^{(j'-j_0)\delta} (2^{-|j-j'|(\frac{n}{2} + N_0)} (2^{j'-j_0}|l|)^{-(n+N_0)})^p \sum_{(\epsilon', k'): Q_{j',k'} \in S_{0,l}} g_{j',k'}^{\epsilon'} \right|^p. \end{aligned}$$

The number of  $k'$  satisfying  $Q_{j',k'} \in S_{0,l}$  is  $2^{n(j'-j_0)}$ . Hence we have

$$\begin{aligned} J_{0,l}^Q &\lesssim (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{jp(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon, k): Q_{j,k} \subset Q} \sum_{j_0 \leq j' \leq j} 2^{(j'-j_0)\delta} \\ &\quad (2^{-|j-j'|(\frac{n}{2} + N_0)} (2^{j'-j_0}|l|)^{-(n+N_0)})^p (2^{j'-j_0})^{n(p-1)} \sum_{(\epsilon', k'): Q_{j',k'} \in S_{0,l}} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim |l|^{-p(N_0-\delta)} \| \{g_{j,k}^{\epsilon}\}_{(\epsilon, j, k) \in \Lambda_n} \|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}}^p. \end{aligned}$$

Case 2:  $j < j'$ . We also divide the argument of this case into two subcases

Subcase 2.1:  $|l| \leq 8^n$ . The proof of this case is the similar to that of Subcase 1.1.

Subcase 2.2:  $|l| > 8^n$ . For this case, we can see that

$$\begin{aligned} |a_{j,k,j',k'}^{\epsilon,\epsilon'}| &\lesssim 2^{-|j-j'|(\frac{n}{2}+N_0)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+\frac{N_0}{2}} \\ &\lesssim 2^{-(j'-j)(\frac{n}{2}+N_0)} 2^{(n+\frac{N_0}{2})(j'-j)} (1 + |2^{j'-j}k - k'|)^{-(n+\frac{N_0}{2})} \\ &\lesssim 2^{(j'-j)(\frac{n}{2}-\frac{N_0}{2})} (1 + |2^{j'-j}k - k'|)^{-(n+\frac{N_0}{2})}. \end{aligned}$$

Let  $x_0$  be the center of  $Q$ . If  $Q_{j',k'} \subset Q$  and  $Q_{j',k'} \subset 2^{-j_0}l + Q$ , we have

$$\begin{aligned} |2^{j'-j}k - k'| &= 2^{j'}|2^{-j}k - x_0 - 2^{-j_0}l + x_0 + 2^{-j_0}l - 2^{-j'}k'| \\ &\geq 2^{j'}(2^{-j_0}|l| - 2^{-j_0} - 2^{-j_0}) \\ &\gtrsim 2^{j'-j_0}|l|, \end{aligned}$$

where we have used the fact that  $|l| > 8^n$ . Hence we can get for any  $\delta > 0$ ,

$$|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \lesssim 2^{(j'-j)(\frac{n}{2}-\frac{N_0}{2})} (2^{j'-j_0}|l|)^{-(n+\frac{N_0}{2})}.$$

By Hölder's inequality, we obtain

$$\begin{aligned} J_{0,l}^Q &\lesssim (1 + |l|)^{n+\delta} |Q|^{\frac{py_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon,k): Q_{j,k} \subset Q} \sum_{j_0 < j \leq j'} 2^{\delta(j'-j_0)} \\ &\quad \left[ 2^{(j'-j)(\frac{n}{2}-\frac{N_0}{2})} (2^{j'-j_0}|l|)^{-(n+\frac{N_0}{2})} \right]^p 2^{n(p-1)(j'-j_0)} \sum_{(\epsilon',k'): Q_{j',k'} \in S_{0,l}} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim (1 + |l|)^{n+\delta-pn-\frac{pN_0}{2}} |Q|^{\frac{py_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon,k): Q_{j,k} \subset Q} \sum_{j_0 < j \leq j'} \\ &\quad 2^{\delta(j'-j_0)} \left[ 2^{(j'-j)(\frac{n}{2}-\frac{N_0}{2})} (2^{j'-j_0}|l|)^{-(n+\frac{N_0}{2})} \right]^p 2^{n(p-1)(j'-j_0)} \sum_{(\epsilon',k'): Q_{j',k'} \in S_{0,l}} |g_{j',k'}^{\epsilon'}|^p. \end{aligned}$$

Notice that the number of  $Q_{j,k} \subset Q$  is  $2^{n(j-j_0)}$ . So, it follows that

$$\begin{aligned} J_{0,l}^Q &\leq (1 + |l|)^{n+\delta-pn-\frac{pN_0}{2}} |Q|^{\frac{py_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{n(j-j_0)} \\ &\quad \sum_{j_0 < j \leq j'} 2^{(j'-j_0)[\delta+n(p-1)-p(n+\frac{N_0}{2})]} 2^{(j'-j)[\frac{p}{2}(n-N_0)]} \sum_{(\epsilon',k'): Q_{j',k'} \in S_{0,l}} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim (1 + |l|)^{n+\delta-pn-\frac{pN_0}{2}} |Q|^{\frac{py_2}{n}-1} \sum_{j \leq j'} 2^{(j'-j)[\frac{p}{2}(n-N_0)-n-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})]} \\ &\quad \sum_{nj' \geq -\log_2 |Q|} 2^{(j'-j_0)[\delta+n(p-1)-p(n+\frac{N_0}{2})+n]} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{(\epsilon',k'): Q_{j',k'} \in S_{0,l}} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim (1 + |l|)^{n+\delta-pn-\frac{pN_0}{2}} \|\{g_{j,k}^{\epsilon}\}_{(\epsilon,j,k) \in \Lambda_n}\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}}. \end{aligned}$$

Next, we consider the case  $\tau > 0$ . For this case, we can see that  $j > j_0$  under  $Q_{j,k} \subset Q$ . On the other hand, because  $\tau \geq 1$ ,  $Q_{j',k'} = 2^{\tau-j_0}l + 2^\tau Q$ , we have  $j' = j_0 - \tau$ . It is easy to see  $j > j'$ . We only need to consider  $|l| \leq 8^n$  and  $|l| > 8^n$  for  $j > j'$ , respectively.

Case 3:  $|l| \leq 8^n$  and  $j > j'$ . Similarly, since  $|j - j'| = j - j_0 + \tau > \tau$ , we can see

$$\begin{aligned} |a_{j,k,j',k'}^{\epsilon,\epsilon'}| &\lesssim 2^{-|j-j'|(\frac{n}{2}+N_0)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N_0} \\ &\lesssim 2^{-|j-j'|(\frac{n+N_0}{2})} 2^{-\frac{|j-j'|N_0}{2}} \\ &\lesssim 2^{-|j-j'|(\frac{n+N_0}{2})} 2^{-\frac{\tau N_0}{2}}. \end{aligned}$$

Note that there is only one dyadic cube  $Q_{j',k'} \in S_{\tau,l}$  for  $\tau \geq 1$ . Thus

$$\begin{aligned} &\sum_{\tau \geq 1} \sum_{l: |l| \leq 8^n} J_{\tau,l}^Q \\ &\lesssim \sum_{\tau \geq 1} 2^{\tau\delta} \sum_{l: |l| \leq 8^n} (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \sum_{(\epsilon,k): Q_{j,k} \subset Q} 2^{-\frac{p\tau N_0}{2}} 2^{-p|j-j'|(\frac{n+N_0}{2})} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim \sum_{\tau \geq 1} 2^{\tau\delta} \sum_{l: |l| \leq 8^n} (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad 2^{n(j-j_0)} 2^{-\frac{p\tau N_0}{2}} 2^{-p|j-j'|(\frac{n+N_0}{2})} |g_{j',k'}^{\epsilon'}|^p \\ &\lesssim \sum_{\tau \geq 1} 2^{\tau\delta} \sum_{l: |l| \leq 8^n} (1 + |l|)^{n+\delta} |Q_\tau|^{1-\frac{p\gamma_2}{n}} |Q|^{\frac{p\gamma_2}{n}-1} \\ &\quad \sum_{j \geq j'} 2^{(j-j')[p\gamma_1 + \frac{pn}{2} - n + n - \frac{pn}{2} - pN_0]} 2^{\tau(n-\frac{pN_0}{2})} \left[ |Q_\tau|^{\frac{p\gamma_2}{n}-1} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} |g_{j',k'}^{\epsilon'}|^p \right] \\ &\lesssim \sum_{\tau \geq 1} 2^{\tau\delta} (2n - p\gamma_2 - \frac{pN_0}{2}) \sum_{l: |l| \leq 8^n} (1 + |l|)^{n+\delta} \|\{g_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

Case 4:  $|l| > 8^n$  and  $j > j'$ . Notice that  $Q_{j,k} \subset Q$  and  $Q_{j',k'} = 2^{\tau-j_0}l + Q_\tau$ . For  $\tau \geq 1$ , we can similarly get

$$\begin{aligned} |2^{j'-j}k - k'| &= 2^{j'}|2^{-j}k - 2^{-j'}k'| \\ &\geq 2^{j'}(2^{\tau-j_0}|l| - |2^{-j}k - x_0| - |x_0 + 2^{\tau-j_0}l - 2^{-j'}k'|) \\ &\gtrsim 2^{\tau+j'-j_0}|l|. \end{aligned}$$

The above estimate implies

$$\begin{aligned} |a_{j,k,j',k'}^{\epsilon,\epsilon'}| &\leq 2^{-|j-j'|(\frac{n}{2}+N_0)} (1 + |2^{j'-j}k - k'|)^{-(n+N_0)} \\ &\lesssim 2^{-|j-j'|(\frac{n}{2}+N_0)} (2^{\tau+j'-j_0}|l|)^{-(n+N_0)} \\ &\lesssim 2^{-|j-j'|(\frac{n+N_0}{2})} 2^{-\frac{\tau N_0}{2}} |l|^{-(n+N_0)}, \end{aligned}$$

where we have used the facts that  $j' = j_0 - \tau$  and  $j > j'$  again. Because for  $\tau \geq 1$  there is only a dyadic cube  $Q_{j',k'}$  in  $S_{\tau,l}$ , we obtain, by  $j' - j_0 = \tau$ ,

$$\begin{aligned}
J_{\tau,l}^Q &\leq 2^{\tau\delta} (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left| \sum_{(\epsilon,k): Q_{jk} \subset Q} \sum_{(\epsilon',j',k'): Q_{j',k'} \in S_{\tau,l}} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'} \right|^p \\
&\lesssim 2^{\tau\delta} (1 + |l|)^{n+\delta} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{nj \geq -\log_2 |Q|} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad 2^{n(j-j_0)} \left[ 2^{-|j-j'| \frac{n+N_0}{2}} 2^{-\frac{\tau N_0}{2}} |l|^{-(n+N_0)} \right]^p |g_{j',k'}^{\epsilon'}|^p \\
&\lesssim 2^{\tau\delta} (1 + |l|)^{n+\delta} |Q_\tau|^{1-\frac{p\gamma_2}{n}} |Q|^{\frac{p\gamma_2}{n}-1} \sum_{j > j'} 2^{p(j-j')(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad 2^{n(j-j_0)} \left[ 2^{-|j-j'| \frac{n+N_0}{2}} 2^{-\frac{\tau N_0}{2}} |l|^{-(n+N_0)} \right]^p \left[ |Q_\tau|^{\frac{p\gamma_2}{n}-1} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} |g_{j',k'}^{\epsilon'}|^p \right] \\
&\lesssim 2^{\tau(\delta+2n-p\gamma_2-\frac{pN_0}{2})} |l|^{n+\delta-pn-pN_0} \\
&\quad \sum_{j > j'} 2^{(j-j')[p\gamma_1 + \frac{pn}{2} - n + n - \frac{p(n+N_0)}{2}]} \|\{g_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

Because  $N_0$  is large enough, we have

$$\sum_{\tau \geq 1} \sum_{|l| > 8^n} J_{\tau,l}^Q \lesssim \|\{g_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.$$

This completes the proof of Theorem 2.5.  $\square$

According to Theorem 2.5, Lemma 2.2 can be described by using the Meyer wavelets via the following Lemma 2.6 whose proof of can be also obtained through [39].

**Lemma 2.6.** *The wavelet characterization in Lemma 2.2 is also true for the Meyer wavelets.*

By applying the fact that the definition of Besov-Q spaces does not depend on the choice of  $\phi$  or by using the fact that the continuity of Calderón-Zygmund operators on the Besov-Q spaces in the above Lemma 2.5, we have

**Corollary 2.7.** *For any  $\frac{1}{2} \leq \lambda \leq 2$ , we have  $\|f(\lambda \cdot)\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}} \approx \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}$ .*

## 2.2. Critical spaces and inclusion relations.

**Definition 2.8.** An initial data space is called critical for (0.1), if it is invariant under the scaling  $f_\lambda(x) = \lambda^{2\beta-1} f(\lambda x)$ .

Note that, if  $u(t, x)$  is a solution of (0.1) and we replace  $u(t, x)$ ,  $p(t, x)$ ,  $a(x)$  by

$$u_\lambda(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x), \quad p_\lambda(t, x) = \lambda^{4\beta-2} u(\lambda^{2\beta} t, \lambda x)$$

and  $a_\lambda(x) = \lambda^{2\beta-1}a(\lambda x)$ , respectively,  $u_\lambda(t, x)$  is also a solution of (0.1). So, the critical spaces occupy a significant place for (0.1). For  $\beta = 1$ ,

$$\begin{cases} \dot{L}_{\frac{n}{2}-1}^2(\mathbb{R}^n) = \dot{B}_2^{-1+\frac{n}{2},2}(\mathbb{R}^n); \\ L^n(\mathbb{R}^n); \\ \dot{B}_p^{-1+\frac{n}{p},\infty}(\mathbb{R}^n), p < \infty; \\ BMO^{-1}(\mathbb{R}^n); \\ \dot{B}_{2,2}^{\alpha-1,\alpha}(\mathbb{R}^n), \end{cases}$$

are critical spaces. For the general  $\beta$ ,

$$\begin{cases} \dot{B}_p^{1+\frac{n}{p}-2\beta,\infty}(\mathbb{R}^n), p < \infty; \\ \dot{B}_{2,2}^{\alpha-\beta+1,\alpha+\beta}(\mathbb{R}^n), \end{cases}$$

are critical spaces.

For  $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ , we have the following dilation-invariance.

**Lemma 2.9.** *For  $\beta > \frac{1}{2}$  and  $\gamma_1 - \gamma_2 = 1 - 2\beta$ , each  $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$  is a critical space, i.e.,*

$$\|\lambda^{\gamma_2-\gamma_1} f(\lambda \cdot)\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}} \approx \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}} \quad \forall \quad \lambda > 0.$$

*Proof.* By Corollary 2.7, we only need to consider the case  $\lambda = 2^w$  and  $w \in \mathbb{Z}$ . For this case, let  $b_{j,k}^{\epsilon,w} = 2^{-\frac{nw}{2}} a_{j-w,k}^\epsilon$ ,  $Q_w = 2^w Q$ , we have:

$$\begin{aligned} \|f(2^w \cdot)\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}} &= |Q|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{nj \geq -\log_2 |Q|} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon,k): Q_{j,k} \subset Q} |b_{j,k}^{\epsilon,w}|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \\ &= |Q|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{nj \geq -\log_2 |Q|} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon,k): Q_{j,k} \subset Q} |2^{-\frac{nw}{2}} a_{j-w,k}^\epsilon|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \\ &= 2^{-nw(\frac{\gamma_2}{n}-\frac{1}{p})} |Q_w|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left[ \sum_{nj \geq -\log_2 |Q_w|} 2^{(j+w)q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\ &\quad \left. \times \left( \sum_{(\epsilon,k): Q_{j,k} \subset Q_w} |2^{-\frac{nw}{2}} a_{j,k}^\epsilon|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}} \\ &= 2^{w(\gamma_1-\gamma_2)} \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

□

To better understand why the Besov-Q spaces are larger than most of the spaces cited in the introduction, we consider now the inclusions among the Q-spaces, the Besov spaces and the Besov-Q spaces. First, we have

**Lemma 2.10.** *Given  $1 < p, q < \infty$  and  $\gamma_1, \gamma_2 \in \mathbb{R}$ .*

(i) *If  $q_1 \leq q_2$ , then  $\dot{B}_{p,q_1}^{\gamma_1,\gamma_2} \subset \dot{B}_{p,q_2}^{\gamma_1,\gamma_2}$ .*

(ii)  *$\dot{B}_{p,q}^{\gamma_1,\gamma_2} \subset \dot{B}_\infty^{\gamma_1-\gamma_2,\infty}(\mathbb{R}^n)$ .*

(iii) *Given  $p_1 \geq 1$ . For  $w = 0, q_1 = 1$  or  $w > 0, 1 \leq q_1 \leq \infty$ , one has  $\dot{B}_{p,q}^{\gamma_1,\gamma_2+w} \subset \dot{B}_{\frac{p}{p_1},\frac{q}{q_1}}^{\gamma_1-w,\gamma_2}$ .*

*Proof.* The inclusion (i) follows from the sequence space inclusion  $l^{q_1} \subset l^{q_2}$  for  $q_1 \leq q_2$ .

For (ii), assume  $f \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}$ . In (2.2), taking  $Q = Q_{j,k}$ , we have

$$\sup_j 2^{-jn(\frac{\gamma_2}{n} - \frac{1}{p})} 2^{j(\gamma_1 + \frac{n}{2} - \frac{n}{p})} |a_{j,k}^\epsilon| < \infty.$$

Equivalently,  $\sup_j 2^{j(\gamma_1 - \gamma_2 + \frac{n}{2})} |a_{j,k}^\epsilon| < \infty$ . By the wavelet characterization of  $\dot{B}_\infty^{\gamma_1 - \gamma_2, \infty}(\mathbb{R}^n)$ , we can see  $f \in \dot{B}_\infty^{\gamma_1 - \gamma_2, \infty}(\mathbb{R}^n)$ .

To prove (iii), let  $0 < p_0, q_0 \leq \infty, p_1 \geq 1$ . For  $w = 0, q_1 = 1$  or  $w > 0, 1 \leq q_1 \leq \infty$ , we have

$$\begin{aligned} & |Q|^{\frac{\gamma_2}{n} - \frac{1}{p_0}} \left( \sum_{nj \geq -\log_2 |Q|} 2^{jq_0(\gamma_1 - w + \frac{n}{2} - \frac{n}{p_0})} \left( \sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^{p_0} \right)^{\frac{q_0}{p_0}} \right)^{\frac{1}{q_0}} \\ & \lesssim |Q|^{\frac{\gamma_2}{n} - \frac{1}{p_1 p_0}} \left( \sum_{nj \geq -\log_2 |Q|} 2^{jq_0(\gamma_1 - w + \frac{n}{2} - \frac{n}{p_1 p_0})} \left( \sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^{p_1 p_0} \right)^{\frac{q_0}{p_1 p_0}} \right)^{\frac{1}{q_0}} \\ & \lesssim |Q|^{\frac{\gamma_2 + w}{n} - \frac{1}{p_1 p_0}} \left( \sum_{nj \geq -\log_2 |Q|} 2^{jq_1 q_0(\gamma_1 + \frac{n}{2} - \frac{n}{p_1 p_0})} \left( \sum_{(\epsilon, k): Q_{j,k} \subset Q} |a_{j,k}^\epsilon|^{p_1 p_0} \right)^{\frac{q_1 q_0}{p_1 p_0}} \right)^{\frac{1}{q_1 q_0}}. \end{aligned}$$

Hence

$$\|f\|_{\dot{B}_{p_0 q_0}^{\gamma_1 - w, \gamma_2}} \leq \|f\|_{\dot{B}_{p_1 p_0 q_1 q_0}^{\gamma_1, \gamma_2 + w}}.$$

□

For  $0 \leq \alpha - \beta + 1$  and  $\alpha + \beta - 1 \leq \frac{n}{2}$ , we say that  $f$  belongs to the Q-type space  $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$  provided

$$\sup_Q r^{2(\alpha + \beta - 1) - n} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n + 2(\alpha - \beta + 1)}} dx dy < \infty,$$

where the supremum is taken over all cubes with sidelength  $r$ . This definition was used in [18] to extend the results in [35] which initiated a PDE-analysis of the original Q-spaces introduced in [7] (cf. [5, 6], [27], [34], and [37] for more information). The following is a direct consequence of Lemmas 2.2, 2.6 and 2.10.

**Corollary 2.11.**

- (i) If  $0 \leq \alpha - \beta + 1 < 1, \alpha + \beta - 1 \leq \frac{n}{2}$ , then  $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n) = \dot{B}_{2,2}^{\alpha - \beta + 1, \alpha + \beta - 1}$ .
- (ii) If  $p = \frac{n}{\gamma_2}$ , then  $\dot{B}_{p,q}^{\gamma_1, \gamma_2} = \dot{B}_p^{\gamma_1, q}(\mathbb{R}^n)$ .
- (iii) Given  $w = 0, v = 1$  or  $w > 0, 1 \leq v \leq \infty$ . If  $p = \frac{n}{\gamma_2 + w}$ , then

$$\dot{B}_p^{\gamma_1, q}(\mathbb{R}^n) \subset \dot{B}_{\frac{n}{u(w + \gamma_2)}, \frac{q}{v}}^{\gamma_1 - w, \gamma_2}.$$

*Remark 2.12.* In [31], J. Wu got the well-posedness of (0.1) with an initial data in the critical Besov space  $\dot{B}_p^{1 + \frac{n}{p} - 2\beta, q}(\mathbb{R}^n)$ . Given  $1 < p_0, q_0 < \infty$ . By

Lemma 2.10, we can see that if  $1 < p \leq p_0$ ,  $1 < q \leq q_0$  and  $\beta > 0$ ,

$$\dot{B}_p^{1+\frac{n}{p}-2\beta, q}(\mathbb{R}^n) \subset \dot{B}_{p_0, q_0}^{1+\frac{n}{p}-2\beta, \frac{n}{p}}.$$

### 3. BESOV-Q SPACES VIA SEMIGROUPS

In this section, we establish a semigroup characterization of the Besov-Q spaces. To do so, recall the following semigroup characterization of  $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$ :

**Lemma 3.1.** ([18]) *Given  $\max\{\alpha, 1/2\} < \beta < 1$  and  $\alpha + \beta - 1 \geq 0$ .  $f \in \mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$  if and only if*

$$\sup_{x \in \mathbb{R}^n \text{ \& } r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} |\nabla e^{-t(-\Delta)^\beta} f(y)|^2 t^{-\frac{\alpha}{\beta}} dy dt < \infty.$$

This characterization was used to derive the global existence and uniqueness of a mild solution to (0.1) with a small initial data in  $\nabla \cdot (\mathcal{Q}_\alpha^\beta(\mathbb{R}^n))^n$ . Notice that

$$\mathcal{Q}_\alpha^\beta(\mathbb{R}^n) = \dot{B}_{2,2}^{\alpha+\beta-1, \alpha-\beta+1}.$$

So, in order to get the corresponding result of (0.1) with a small initial data in  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$  under  $|p-2|+|q-2| \neq 0$ , we need a more meticulous relation among time, frequency and locality. For this purpose, by the Meyer wavelets and the fractional heat semigroups, we introduce some new tent spaces associated with  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , and then establish some a connection between these tent spaces and  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ .

**3.1. Wavelets and semigroups.** We choose  $\phi$  in the following lemma which is a variation of Lemma 1.1 in [9] or a variation of the construction of Meyer's wavelets (Meyer calls them the Littlewood-Paley wavelets in Section 2 of Chapter 3 in [23])

**Lemma 3.2.** *Fix  $\beta > 0$ . There exist a constant  $C_\beta > 0$  and a radial function  $\phi \in S(\mathbb{R}^n)$  such that*

- (i)  $\int_{\mathbb{R}^n} x^\gamma \phi(x) dx = 0, \forall \gamma \in \mathbb{N}^n$ ;
- (ii)  $\int_0^\infty (\hat{\phi}(t^{\frac{1}{2\beta}} \xi))^2 \frac{dt}{t} = 1, \forall \xi \neq 0$ ;
- (iii)  $C_\beta \int_0^\infty \hat{\phi}(t^{\frac{1}{2\beta}} \xi) e^{-t} \frac{dt}{t} = 1$ .

For  $\beta > 0$ , let  $\hat{K}_t^\beta(\xi) = e^{-t|\xi|^{2\beta}}$  and  $\phi_t^\beta(x) = t^{-\frac{n}{2\beta}} \phi(t^{-\frac{1}{2\beta}} x)$  with the Fourier transform  $\hat{\phi}_t^\beta(\xi) = \hat{\phi}(t^{\frac{1}{2\beta}} \xi)$ . We have

$$f(t, x) = e^{-t(-\Delta)^\beta} f(x) = K_t^\beta * f(x).$$

Observe that

$$(3.1) \quad \hat{f}(\xi) = C_\beta \int_0^\infty \Phi(t^{\frac{1}{2\beta}} \xi) e^{-t} \frac{dt}{t} \hat{f}(\xi) = C_\beta \int_0^\infty \Phi(t^{\frac{1}{2\beta}} \xi) e^{-t|\xi|^{2\beta}} \hat{f}(\xi) \frac{dt}{t}.$$

Hence, by (3.1),

$$(3.2) \quad f(x) = C_\beta \int_0^\infty f(t, \cdot) * \phi_t^\beta(x) \frac{dt}{t} := \pi_\phi f(\cdot, x).$$

For the Meyer wavelets  $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$ , let  $a_{j,k}^\epsilon(t) = \langle f(t, \cdot), \Phi_{j,k}^\epsilon \rangle$  and  $a_{j,k}^\epsilon = \langle f, \Phi_{j,k}^\epsilon \rangle$ . By Lemma 1.1, we get

$$f(x) = \sum_{\epsilon,j,k} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon(x) \quad \text{and} \quad f(t, x) = \sum_{\epsilon,j,k} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x).$$

If  $f(t, x) = K_t^\beta * f(x)$ , then

$$\begin{aligned} a_{j,k}^\epsilon(t) &= \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \langle K_t^\beta \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ (3.3) \quad &= \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \int e^{-t|\xi|^{2\beta}} \hat{\Phi}^{\epsilon'}(2^{-j'} \xi) \hat{\Phi}^\epsilon(2^{-j} \xi) e^{-i(2^{-j'} k' - 2^{-j} k) \xi} d\xi \\ &= \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \int e^{-t2^{2j\beta} |\xi|^{2\beta}} \hat{\Phi}^{\epsilon'}(2^{j-j'} \xi) \hat{\Phi}^\epsilon(\xi) e^{-i(2^{j-j'} k' - k) \xi} d\xi. \end{aligned}$$

By (3.3), we have

**Lemma 3.3.** *Let  $\{\Phi_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$  be Meyer wavelets. For the fixed  $\beta > 0$ , there exist a constant  $N_\beta$  large enough and a fixed small constant  $\tilde{c} > 0$  such that if  $N > N_\beta$  then*

$$(3.4) \quad |a_{j,k}^\epsilon(t)| \lesssim e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N} \quad \forall \quad t2^{2\beta j} \geq 1$$

and

$$(3.5) \quad |a_{j,k}^\epsilon(t)| \lesssim \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N} \quad \forall \quad 0 \leq t2^{2\beta j} \leq 1.$$

*Proof.* Formally, we can write

$$\begin{aligned} a_{j,k}^\epsilon(t) &= \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \langle K_t^\beta * \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \langle e^{-t(-\Delta)^\beta} \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'} \langle (-\Delta)^{m\beta} \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle. \end{aligned}$$



For the integer  $m$ , let  $\widehat{\Phi^{\epsilon,m,\beta}}(\xi) = |\xi|^{2m\beta} \widehat{\Phi^\epsilon}(\xi)$ . Then, by Plancherel's equality, one has

$$\begin{aligned}
& \langle (-\Delta)^{m\beta} \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\
&= \int |\xi|^{2m\beta} \widehat{\Phi^{\epsilon'}}(2^{-j'} \xi) \widehat{\Phi^\epsilon}(2^{-j} \xi) e^{-i(2^{-j'} k' - 2^{-j} k) \xi} d\xi \\
&= \int 2^{2mj\beta} |\xi|^{2m\beta} \widehat{\Phi^{\epsilon'}}(2^{j-j'} \xi) \widehat{\Phi^\epsilon}(\xi) e^{-i(2^{j-j'} k' - k) \xi} d\xi \\
&= \frac{2^{2mj\beta}}{(1 + |2^{j-j'} k' - k|)^N} \int \widehat{\Phi^{\epsilon'}}(2^{j-j'} \xi) \widehat{\Phi^{\epsilon,m,\beta}}(\xi) (1 + \frac{1}{i} \partial_\xi)^N e^{-i(2^{j-j'} k' - k) \xi} d\xi.
\end{aligned}$$

Because  $|j - j'| \leq 1$  and the support of the wavelet function  $\widehat{\Phi^\epsilon}$  is contained in  $\{\xi \in \mathbb{R}^n, \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3} \sqrt{n}\}$ . Using the integration by parts, we can get

$$\left| \langle (-\Delta)^{m\beta} \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \right| \leq 2^{2mj\beta} (1 + |2^{j-j'} k' - k|)^{-N}.$$

Finally,

$$\begin{aligned}
|a_{j,k}^\epsilon(t)| &\leq \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| 2^{2mj\beta} (1 + |2^{j-j'} k' - k|)^{-N} \\
&\lesssim e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N}.
\end{aligned}$$

This completes the proof of Lemma 3.3.  $\square$

Conversely, by the reproducing formula (3.2),  $a_{j,k}^\epsilon$  can be also expressed by  $\{a_{j',k'}^{\epsilon'}(t)\}$  as follows.

$$(3.6) \quad a_{j,k}^\epsilon = \int_{\mathbb{R}_+^{1+n}} \sum_{\epsilon', j', k'} a_{j',k'}^{\epsilon'}(t) (\phi_t^\beta * \Phi_{j',k'}^{\epsilon'}(x)) \Phi_{j,k}^\epsilon(x) \frac{dx dt}{t}.$$

An integration by parts yields the following estimate.

**Lemma 3.4.** *Let  $\{a_{j,k}^\epsilon\}_{(\epsilon,j,k) \in \Lambda_n}$  be defined in (3.6). For any  $N > 0$ ,*

$$(3.7) \quad |a_{j,k}^\epsilon| \leq \sum_{|j-j'| \leq 1} \int_0^\infty (\max\{t2^{2j'\beta}, t^{-1}2^{-2j'\beta}\})^{-N} \left( \sum_{\epsilon', k'} \frac{|a_{j',k'}^{\epsilon'}(t)|}{(1 + |2^{j-j'} k' - k|)^N} \right) \frac{dt}{t}.$$

*Proof.* Case 1:  $t2^{2j'\beta} \geq 1$ . For this case,  $\max\{t2^{2j'\beta}, t^{-1}2^{-2j'\beta}\} = t2^{2j'\beta}$ . By (3.6), we have

$$\begin{aligned}
a_{j,k}^\epsilon &= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} \phi_t^\beta * \Phi_{j',k'}^{\epsilon'}(x) \Phi_{j,k}^\epsilon(x) dx \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} \widehat{\phi}(t^{\frac{1}{2\beta}} \xi) \widehat{\Phi}^{\epsilon'}(2^{-j'} \xi) \widehat{\Phi}^\epsilon(2^{-j} \xi) e^{-i(2^{-j}k - 2^{-j'}k')\xi} d\xi \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) (1 + |2^{j-j'}k' - k|)^{-N} \int_{\mathbb{R}^n} \widehat{\phi}(2^{j'} t^{\frac{1}{2\beta}} \xi) \widehat{\Phi}^{\epsilon'}(\xi) \widehat{\Phi}^\epsilon(2^{j'-j} \xi) \\
&\quad [(1 + \frac{1}{t} \partial_\xi)^N e^{-i(2^{j-j'}k' - k)\xi}] d\xi \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) (1 + t^{\frac{1}{2\beta}} 2^{j'})^N (1 + |2^{j-j'}k' - k|)^{-N} \\
&\quad \int_{\mathbb{R}^n} \sum_{l=0}^N C_N^l (\partial_\xi^l \widehat{\phi})(t^{\frac{1}{2\beta}} 2^{j'} \xi) \partial_\xi^{N-l} [\widehat{\Phi}^{\epsilon'}(\xi) \widehat{\Phi}^\epsilon(2^{j'-j} \xi)] e^{-i(2^{j-j'}k' - k)\xi} d\xi \frac{dt}{t}.
\end{aligned}$$

Because  $\phi \in \mathcal{S}(\mathbb{R}^n)$ , then for any  $N' > 0$ ,

$$\left| (\partial_\xi^l \widehat{\phi})(t^{\frac{1}{2\beta}} 2^{j'} \xi) \right| \lesssim \frac{(t^{\frac{1}{2\beta}} 2^{j'})^l}{[1 + (t^{\frac{1}{2\beta}} 2^{j'} |\xi|)]^{N'}}.$$

On the other hand, if  $\xi \in B(0, \frac{2}{3}\pi)$ , then  $\widehat{\Phi}^{\epsilon'}(\xi) = 0$ . We can obtain that

$$\begin{aligned}
|a_{j,k}^\epsilon| &\leq \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}(t)|}{(1 + |2^{j-j'}k' - k|)^N} (1 + t^{\frac{1}{2\beta}} 2^{j'})^N (1 + t^{\frac{1}{2\beta}} 2^{j'})^{-N'} \frac{dt}{t} \\
&\leq \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}(t)| (1 + |2^{j-j'}k' - k|)^{-N} (t2^{2j'\beta})^{-N'} \frac{dt}{t}.
\end{aligned}$$

holds for  $N'$  large enough.

Case 2:  $t2^{2j'\beta} \leq 1$ . For this case,  $\max\{t2^{2j'\beta}, t^{-1}2^{-2j'\beta}\} = t^{-1}2^{-2j'\beta}$ . By (3.6), we have

$$\begin{aligned}
a_{j,k}^\epsilon &= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} \phi_t^\beta * \Phi_{j',k'}^{\epsilon'}(x) \Phi_{j,k}^\epsilon(x) dx \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} \widehat{\phi}(t^{\frac{1}{2\beta}} \xi) \widehat{\Phi}^{\epsilon'}(2^{-j'} \xi) \widehat{\Phi}^\epsilon(2^{-j} \xi) e^{-i(2^{-j}k - 2^{-j'}k')\xi} d\xi \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) (|2^{j-j'}k' - k|)^{-N} \int_{\mathbb{R}^n} \widehat{\phi}(2^{j'} t^{\frac{1}{2\beta}} \xi) \widehat{\Phi}^{\epsilon'}(\xi) \widehat{\Phi}^\epsilon(2^{j'-j} \xi) \\
&\quad [(\frac{1}{t} \partial_\xi)^N e^{-i(2^{j-j'}k' - k)\xi}] d\xi \frac{dt}{t} \\
&= \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} a_{j',k'}^{\epsilon'}(t) (t^{\frac{1}{2\beta}} 2^{j'})^N (|2^{j-j'}k' - k|)^{-N} \\
&\quad \int_{\mathbb{R}^n} \sum_{l=0}^N C_N^l (\partial_\xi^l \widehat{\phi})(t^{\frac{1}{2\beta}} 2^{j'} \xi) \partial_\xi^{N-l} [\widehat{\Phi}^{\epsilon'}(\xi) \widehat{\Phi}^\epsilon(2^{j'-j} \xi)] e^{-i(2^{j-j'}k' - k)\xi} d\xi \frac{dt}{t}.
\end{aligned}$$

Similar to Case 1, we can get

$$|a_{j,k}^\epsilon| \lesssim \int_0^\infty \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}(t)| (t^{\frac{1}{2\beta}} 2^{j'})^{N'} (1 + |2^{j-j'}k' - k|)^{-N} \frac{dt}{t}.$$

Taking  $\frac{N'}{2\beta} > N$ , we obtain the desired estimate.  $\square$

**3.2. Tent spaces generated by Besov-Q spaces.** In this subsection, on  $\mathbb{R}_+^{1+n}$ , we introduce a new tent type space  $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$  associated with  $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$ , and then establish a relation between  $\dot{B}_{p,q}^{\gamma_1,\gamma_2}$  and  $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$  via the fractional heat semigroup  $e^{-t(-\Delta)^\beta}$ .

Given a function  $a(\cdot, \cdot)$  on  $\mathbb{R}_+^{1+n}$  of the form

$$a(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x).$$

For dyadic cube  $Q_r$  and  $m \in \mathbb{R}$ , let

$$\begin{cases} I_{p,Q_r,m}(t) = \sum_{(\epsilon, j, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p (t^{2j\beta})^m; \\ I_{p,Q_r}(t) = I_{p,Q_r,0}(t), \end{cases}$$

and write

$$\begin{cases} I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, \frac{-\log_2 t}{2\beta}\}} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_{p,Q_r,m}(t))^{\frac{q}{p}}; \\ I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < \frac{-\log_2 t}{2\beta}} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_{p,Q_r}(t))^{\frac{q}{p}}; \\ I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j\beta}}^{2^{\beta}} I_{p,Q_r,m}(t) \frac{dt}{t} \right)^{\frac{q}{p}}; \\ II_{p,q,Q_r,m}^{\gamma_1,\gamma_2} = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_0^{2^{-2j\beta}} I_{p,Q_r,m}(t) \frac{dt}{t} \right)^{\frac{q}{p}}. \end{cases}$$

The tent type spaces  $\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$  are defined as follows.

**Definition 3.5.** For  $\gamma_1, \gamma_2, m \in \mathbb{R}$ ,  $m' > 0$ ,  $1 < p, q < \infty$  and the above notations on  $a(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x)$ , let

$$\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} = \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I} \cap \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II} \cap \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III} \cap \mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV},$$

where

$$\begin{cases} f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I} \text{ provided } \sup_{t \geq 0} \sup_{x_0, r} I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) < \infty; \\ f \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II} \text{ provided } \sup_{t \geq 0} \sup_{x_0, r} I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) < \infty; \\ f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III} \text{ provided } \sup_{x_0, r} I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} < \infty; \\ f \in \mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV} \text{ provided } \sup_{x_0, r} II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} < \infty. \end{cases}$$

To continue our discussion, we need to introduce two more function spaces  $\mathbb{B}_{\tau,\infty}^\gamma$  and  $\mathbb{B}_{\tau,\infty}^{\gamma_1}$ .

**Definition 3.6.** For  $(\epsilon, j, k) \in \Lambda_n$ , write  $a_{j,k}^\epsilon(t) = \langle a(t, \cdot), \Phi_{j,k}^\epsilon(\cdot) \rangle$ . Given  $\tau > 0$  and  $\gamma \in \mathbb{R}$ . We say that

$$\begin{cases} a(\cdot, \cdot) \in \mathbb{B}_{\tau, \infty}^\gamma & \text{if } \sup_{t2^{2j\beta} \geq 1} (t2^{2j\beta})^\tau 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| + \sup_{0 < t2^{2j\beta} < 1} 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| < \infty; \\ a(\cdot, \cdot) \in \mathbb{B}_{0, \infty}^\gamma & \text{if } t^{\frac{\gamma}{2\beta}} 2^{\frac{nj}{2}} |\langle a(t, \cdot), \Phi_{j,k}^0 \rangle| < \infty. \end{cases}$$

The following inclusions are nontrivial.

**Lemma 3.7.** Given  $1 < p, q < \infty$ ,  $\gamma_1, \gamma_2 \in \mathbb{R}$ ,  $m > p$  and  $m', \tau > 0$ .

(i) If  $m > 0$ , then  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2} \subset \mathbb{B}_{p, \infty}^{\gamma_1 - \gamma_2}$ .

(ii) If  $-2\beta\tau < \gamma < 0 < \beta$ , then  $\mathbb{B}_{\tau, \infty}^\gamma \subset \mathbb{B}_{0, \infty}^\gamma$ .

*Proof.* (i) Without loss of generality, we may assume  $\|f\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}} = 1$ . Under this assumption, we have  $f \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I} \cap \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ . Note that if  $a_{j,k}^\epsilon(t) = \langle f(t, \cdot), \Phi_{j,k}^\epsilon(\cdot) \rangle$  then one can write

$$f(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x).$$

Case 1:  $t2^{2j\beta} \geq 1$ . For this case, one has  $j \geq -\frac{\log_2 t}{2\beta}$ . Because  $f \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ , one gets

$$|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 t, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k): Q_{j,k} \subset Q_r} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \lesssim 1.$$

Taking  $Q_r = Q_{j,k}$  and  $r = 2^{-j}$ , we get

$$2^{-jn(\frac{q\gamma_2}{n} - \frac{q}{p})} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \lesssim 1,$$

that is,

$$|a_{j,k}^\epsilon(t)| \lesssim 2^{j(\gamma_2 - \gamma_1) - \frac{n}{2}} (t2^{2j\beta})^{-\frac{m}{p}}.$$

Case 2:  $t2^{2j\beta} < 1$ . Because  $f \in \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ , one has  $2^{j(\gamma_2 - \gamma_1 - \frac{n}{2})} |a_{j,k}^\epsilon(t)| \lesssim 1$ . This implies  $f \in \mathbb{B}_{p, \infty}^{\gamma_1 - \gamma_2}$ .

(ii) Similarly, we may assume  $\|f\|_{\mathbb{B}_{\tau, \infty}^\gamma} = 1$ . Suppose  $a \in \mathbb{B}_{\tau, \infty}^\gamma$ . Then

$$\sup_{t2^{2j\beta} \geq 1} (t2^{2j\beta})^\tau 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| + \sup_{0 < t2^{2j\beta} < 1} 2^{\frac{nj}{2}} 2^{j\gamma} |a_{j,k}^\epsilon(t)| < \infty.$$

By definition,

$$\begin{aligned} \langle a(t, \cdot), \Phi_{j,k}^0(\cdot) \rangle &= \left\langle \sum_{j' \leq j} a_{j',k'}^{\epsilon'}(t) \Phi_{j',k'}^{\epsilon'}(\cdot), \Phi_{j,k}^0(\cdot) \right\rangle \\ &= \sum_{j' \leq j} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} \Phi_{j',k'}^{\epsilon'}(x) \Phi_{j,k}^0(x) dx \\ &= \sum_{j' \leq j} a_{j',k'}^{\epsilon'}(t) \int_{\mathbb{R}^n} 2^{-\frac{n(j+j')}{2}} \widehat{\Phi}^{\epsilon'}(2^{-j'} \xi) \widehat{\Phi}^0(2^{-j} \xi) d\xi \\ &= \sum_{j' \leq j} a_{j',k'}^{\epsilon'}(t) 2^{\frac{n(j'-j)}{2}} \int_{\mathbb{R}^n} \widehat{\Phi}^{\epsilon'}(\xi) \widehat{\Phi}^0(2^{j'-j} \xi) d\xi. \end{aligned}$$

The rest of the proof is divided into two cases.

Case 1:  $t2^{2j\beta} \leq 1$ . For this case,  $j' \leq j$  implies that  $t2^{2j'\beta} \leq 1$ . Because  $a \in \mathbb{B}_{\tau, \infty}^\gamma$ , we can get  $|a_{j',k'}^{\epsilon'}(t)| \lesssim 2^{-\frac{n_{j'}}{2}} 2^{-j'\gamma}$ . Hence for  $\gamma < 0$ ,

$$\begin{aligned} \left| \langle a(t, \cdot), \Phi_{j,k}^0(\cdot) \rangle \right| &\lesssim \sum_{j' \leq j} 2^{-\frac{n_{j'}}{2}} 2^{-j'\gamma} 2^{\frac{n(j'-j)}{2}} \\ &\lesssim 2^{-\frac{n_j}{2}} \sum_{j' \leq j} 2^{-j'\gamma} \\ &\lesssim 2^{-\frac{n_j}{2}} t^{\frac{\gamma}{2\beta}}, \end{aligned}$$

where we have used both  $t2^{2j\beta} \leq 1$  and  $\gamma < 0$ .

Case 2:  $t2^{2j\beta} > 1$ . As above, we can get

$$\begin{aligned} \left| \langle a(t, \cdot), \Phi_{j,k}^0(\cdot) \rangle \right| &\leq \sum_{j' \leq j} |a_{j',k'}^{\epsilon'}(t)| 2^{\frac{n(j'-j)}{2}} \left| \int_{\mathbb{R}^n} \widehat{\Phi^{\epsilon'}}(\xi) \widehat{\Phi^0}(2^{j'-j}\xi) d\xi \right| \\ &\leq \sum_{-\frac{\log_2 t}{2\beta} < j' \leq j} |a_{j',k'}^{\epsilon'}(t)| 2^{\frac{n(j'-j)}{2}} \left| \int_{\mathbb{R}^n} \widehat{\Phi^{\epsilon'}}(\xi) \widehat{\Phi^0}(2^{j'-j}\xi) d\xi \right| \\ &\quad + \sum_{j' \leq -\frac{\log_2 t}{2\beta}} |a_{j',k'}^{\epsilon'}(t)| 2^{\frac{n(j'-j)}{2}} \left| \int_{\mathbb{R}^n} \widehat{\Phi^{\epsilon'}}(\xi) \widehat{\Phi^0}(2^{j'-j}\xi) d\xi \right| \\ &=: M_1 + M_2. \end{aligned}$$

For  $M_1$ , thanks to  $j' > -\frac{\log_2 t}{2\beta}$ , we have  $|a_{j',k'}^{\epsilon'}(t)| \lesssim (t2^{2j'\beta})^{-\tau} 2^{-\frac{n_{j'}}{2}} 2^{-j'\gamma}$  and

$$\begin{aligned} M_1 &\lesssim \sum_{-\frac{\log_2 t}{2\beta} < j' \leq j} (t2^{2j'\beta})^{-\tau} 2^{-\frac{n_{j'}}{2}} 2^{-j'\gamma} 2^{\frac{n(j'-j)}{2}} \\ &\lesssim 2^{-\frac{n_j}{2}} t^{-\tau} \sum_{-\frac{\log_2 t}{2\beta} < j' \leq j} 2^{-j'(2\beta\tau+\gamma)} \\ &\lesssim 2^{-\frac{n_j}{2}} t^{\frac{\gamma}{2\beta}}. \end{aligned}$$

For  $M_2$ , owing to  $j' \leq -\frac{\log_2 t}{2\beta}$  and  $\gamma < 0$ , we have

$$M_2 \lesssim \sum_{j' \leq -\frac{\log_2 t}{2\beta}} 2^{-\frac{n_{j'}}{2}} 2^{-j'\gamma} 2^{\frac{n(j'-j)}{2}} \lesssim 2^{-\frac{n_j}{2}} t^{\frac{\gamma}{2\beta}}.$$

This completes the proof of (ii).  $\square$

For any dyadic cube  $Q_{j_0,k_0}$ , we always use  $\widetilde{Q}_{j_0,k_0}$  to denote the dyadic cube containing  $Q_{j_0,k_0}$  with side length  $2^{8-j_0}$ .

Given  $(\epsilon, j, k) \in \Lambda_n$ . If  $\epsilon \in E_n$  and  $Q_{j,k} \subset Q_{j_0,k_0}$ , we write  $(\epsilon, k) \in S_{j_0,k_0}^j$ .

For any  $w \in \mathbb{Z}^n$ , denote  $\widetilde{Q}_{j_0,k_0}^w = 2^{8-j_0}w + \widetilde{Q}_{j_0,k_0}$ . Denote  $(\epsilon, k) \in S_{j_0,k_0}^{w,j}$  whenever  $Q_{j,k} \subset \widetilde{Q}_{j_0,k_0}^w$ .

In the rest of this paper, we frequently utilize the so-called  $\alpha$ -triangle inequality below:

$$(a+b)^\alpha \leq a^\alpha + b^\alpha \quad \forall \quad (\alpha, a, b) \in (0, 1] \times (0, \infty) \times (0, \infty).$$

Now we characterize the Besov-Q spaces by using a semigroup operator.

**Theorem 3.8.** *Given  $1 < p < m < \infty$ ,  $m' > 0$ ,  $1 < q < \infty$ ,  $\gamma_1 - \gamma_2 < 0 < \beta$ .  
 (i) If  $f \in \dot{B}_{p,q}^{\gamma_1, \gamma_2}$ , then  $f * K_t^\beta \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ .  
 (ii) The operator  $\pi_\phi$  is a bounded and surjective operator from  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$  to  $\dot{B}_{p,q}^{\gamma_1, \gamma_2}$ .*

*Proof.* (i) We prove that

$$f = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in \dot{B}_{p,q}^{\gamma_1, \gamma_2} \implies f * K_t^\beta = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon \in \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}.$$

via handling four situations.

**Situation 1:**  $K_t^\beta * f \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ . For  $t2^{2j\beta} > 1, m > 0$ , by (3.4), there exists a constant  $N$  large enough such that

$$|a_{j,k}^\epsilon(t)| \lesssim e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1+|2^{j-j'}k' - k|)^N} \lesssim 2^{\frac{-nj}{2}} 2^{j(\gamma_2 - \gamma_1)} e^{-\tilde{c}t2^{2j\beta}}.$$

Choosing a sufficiently large  $N'$  (depending on  $N$ ) in the last estimate, we have

$$\begin{aligned} I_{p,q,Q_r,m}^{\gamma_1, \gamma_2}(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \left[ \sum_{(\epsilon, k) \in S_r^j} (e^{-\tilde{c}t2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N})^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^{\frac{mq}{p}} \\ &\quad \left[ \sum_{(\epsilon, k) \in S_r^j} \left( \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \right)^p \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \left[ \sum_{(\epsilon, k) \in S_r^j} \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'}k' - k|)^{-N} \right]^{\frac{q}{p}}, \end{aligned}$$

where  $p > 1$  has been used. In the sequel, we divide the proof into two cases.

Case 1.1:  $q \leq p$ . Because  $|j - j'| \leq 1$  and  $j > -\log_2 r$ , one gets  $2^{-(j'+1)n} \leq r^n$ . Hence

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) &\lesssim \sum_{w \in \mathbb{Z}^n, |w| \leq 2^n} (1 + |w|)^{-\frac{Nq}{p}} \left\{ |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \right. \\
&\quad \left. 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right]^{\frac{q}{p}} \right\} \\
&+ \sum_{w \in \mathbb{Z}^n, |w| > 2^n} (1 + |w|)^{-\frac{Nq}{p}} \left\{ |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \right. \\
&\quad \left. 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p (2^{nj'} |Q|)^{-N'} \right]^{\frac{q}{p}} \right\} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

Case 1.2:  $q > p$ . Applying Hölder's inequality to  $w \in \mathbb{Z}^n$ , we similarly have

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \left\{ |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right]^{\frac{q}{p}} \right\} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

**Situation 2:**  $K_t^\beta * f \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ . For  $t2^{2\beta j} \leq 1$  and  $m' > 0$ , by (3.5), there exists a natural number  $N$  large enough such that  $N > 2n$  and

$$|a_{j,k}^\epsilon(t)| \lesssim \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N}.$$

Consequently, we have

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j \leq -\frac{\log_2 t}{2\beta}} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \sum_{(\epsilon, k) \in S_r^j} \left( \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N} \right)^p \right]^{\frac{q}{p}}.
\end{aligned}$$

Case 2.1:  $q \leq p$ . We have

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{|w| \leq 2^n} (1 + |w|)^{-\frac{qN}{p}} \\
&\quad \sum_{-\log_2 r - 1 \leq j' \leq -\frac{\log_2 t}{2\beta} - 1} 2^{j'q(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon', k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\
&+ |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{|w| > 2^n} (1 + |w|)^{-\frac{qN}{p}} \\
&\quad \sum_{-\log_2 r - 1 \leq j' \leq -\frac{\log_2 t}{2\beta} - 1} 2^{j'q(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon', k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p (2^{nj'} |Q|)^{-N} \right)^{\frac{q}{p}} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

Case 2.2:  $q > p$ . By Hölder's inequality, we obtain

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r-1 \leq j' \leq -\frac{\log_2 t}{2\beta}-1} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \sum_{(\epsilon',k') \in S_{Q_r}^{w,j'}} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'}k' - k|)^{-N'} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r-1 \leq j' \leq -\frac{\log_2 t}{2\beta}-1} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \sum_{|w| \leq 2^n} (1 + |w|)^{-N'} \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right. \\
&\quad \left. + \sum_{|w| > 2^n} (1 + |w|)^{-N'} (2^{nj'}|Q|)^{-N'} \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right]^{\frac{q}{p}} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

**Situation 3:**  $K_t^\beta * f \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}$ . For this case, we have  $2^{-2j\beta} < t < r^{2\beta}$  and thus

$$|a_{j,k}^\epsilon(t)| \lesssim e^{-ct2^{2j\beta}} \sum_{\epsilon', |j-j'| \leq 1, k'} \frac{|a_{j',k'}^{\epsilon'}|}{(1 + |2^{j-j'}k' - k|)^{N'}} \lesssim 2^{-\frac{n}{2}j} 2^{j(\gamma_2-\gamma_1)} (t2^{2j\beta})^{-\tau}.$$

This yields

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j\beta})^m \right. \\
&\quad \left. \sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} \left( \sum_{\epsilon', |j-j'| \leq 3, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \right)^p \frac{dt}{t} \right]^{\frac{q}{p}}
\end{aligned}$$

Let

$$A_j = \left( \sum_{\epsilon', |j-j'| \leq 3, k'} |a_{j',k'}^{\epsilon'}| (1 + |2^{j-j'}k' - k|)^{-N} \right)^p.$$

Notice that  $j \sim j'$  and the number of  $\epsilon'$  is finite. Applying Hölder's inequality on  $k'$ , we obtain

$$A_j \lesssim \sum_{|j-j'| \leq 3} \sum_{\epsilon', k'} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'}k' - k|)^{-N}.$$

Let  $Q_{j,k}$  and  $Q_{j',k'}$  be two dyadic cubes. Denote by  $\widetilde{Q}_{j,k}$  the dyadic cube containing  $Q_{j,k}$  with side length  $2^{8-j}$ . For  $w \in \mathbb{Z}^n$ , denote by  $Q_{j,k}^w$  the cube  $\widetilde{Q}_{j,k} + 2^{8-j}w$ . It is easy to see that if  $Q_{j',k'} \subset Q_{j,k}^w$ , then

$$(1 + |2^{j-j'}k' - k|)^{-N} \lesssim (1 + |w|)^{-N}$$



(see also (4.2)). We obtain that

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t^{2j\beta})^m \right. \\
&\quad \left. \sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} \sum_{|j-j'| \leq 3} \sum_{\epsilon',k'} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'} k' - k|)^{-N'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t^{2j\beta})^m \sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} \right. \\
&\quad \left. \sum_{|j-j'| \leq 3} \sum_{w \in \mathbb{Z}^n} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |a_{j',k'}^{\epsilon'}|^p (1 + |2^{j-j'} k' - k|)^{-N'} \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

The number of  $Q_{j',k'}$  which are contained in the dyadic cube  $Q_{j,k}^w = 2^{8-j}w + \widetilde{Q}_{j,k}$  equals to  $2^{n(8+j'-j)}$ . On the other hand, for any dyadic cube  $Q_r$  with radius  $r$ , the number of  $Q_{j,k} \subset Q_r$  equals to  $(2^j r)^n$ . Then the number of  $Q_{j',k'}$  which are contained in the dyadic cube  $Q_r^w$  equals  $(2^{8+j'} r)^n$ . Denote  $S_r^{w,j'}$  the set of  $(\epsilon', k')$  such that  $Q_{j',k'} \subset Q_r^w$ . Finally we have

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r-3} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} (t^{2j'\beta}) \sum_{|w| \leq 2^n} (1 + |w|)^{-N} \right. \\
&\quad \left. \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-ct2^{2j'\beta}} |a_{j',k'}^{\epsilon'}|^p \frac{dt}{t} \right]^{\frac{q}{p}} \\
&+ |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r-3} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} (t^{2j'\beta}) \sum_{|w| > 2^n} (1 + |w|)^{-N} \right. \\
&\quad \left. \sum_{(\epsilon',k') \in S_r^{w,j'}} e^{-ct2^{2j'\beta}} |a_{j',k'}^{\epsilon'}|^p (2^{nj'} |Q|)^{-N'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&=: M_1 + M_2.
\end{aligned}$$

Because  $f \in \dot{B}_{p,q}^{\gamma_1,\gamma_2}$ , from

$$\int_{2^{-2j\beta}}^{r^{2\beta}} (t^{2j\beta})^m e^{-cpt2^{2j\beta}} \lesssim 1$$

it follows that that

$$\begin{aligned}
M_1 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r-3} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left( \int_{2^{-2j'\beta}}^{r^{2\beta}} (t^{2j'\beta}) e^{-cpt2^{2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \frac{dt}{t} \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r-3} 2^{j'q(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}.
\end{aligned}$$

For the term  $M_2$ , we divide the estimate into two cases.

Case 3.1:  $q \leq p$ . For this case,  $j' \geq -\log_2 r - 3$  implies  $(2^{nj'} r^n)^{N'} \lesssim 1$  and

$$\begin{aligned}
M_2 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r - 3} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN'}{p}} \\
&\quad \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{(\epsilon', k') \in S_r^{w, j'}} e^{-cpt2^{2j'\beta}} |a_{j', k'}^\epsilon|^p (2^{nj'} |Q|)^{-N'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r - 3} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN'}{p}} \\
&\quad \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{(\epsilon', k') \in S_r^{w, j'}} e^{-cpt2^{2j'\beta}} |a_{j', k'}^\epsilon|^p \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1, \gamma_2}}.
\end{aligned}$$

Case 3.2:  $q > p$ . For this case, by Hölder's inequality and  $j \sim j'$ , we obtain

$$\begin{aligned}
&\left[ \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j\beta})^m \sum_{|w| > 2^n} \sum_{(\epsilon', k') \in S_r^{w, j'}} e^{-cpt2^{2j\beta}} (1 + |w|)^{-N} |a_{j', k'}^{\epsilon'}|^p (2^{nj'} |Q|)^{-N'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \sum_{|w| > 2^n} (1 + |w|)^{-\frac{qN'}{p}} \left( \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j'\beta})^m \sum_{(\epsilon', k') \in S_r^{w, j'}} e^{-cpt2^{2j'\beta}} |a_{j', k'}^{\epsilon'}|^p \frac{dt}{t} \right)^{\frac{q}{p}}.
\end{aligned}$$

The rest of the argument is similar to that of Case 3.1, and so omitted.

**Situation 4:**  $K_t^\beta * f \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, IV}$ . As before, we have

$$\begin{aligned}
II_{p,q,Q_r,m'}^{\gamma_1, \gamma_2} &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} \right. \\
&\quad \left. \left( \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j', k'}^{\epsilon'}| (1 + |2^{j-j'} k' - k|)^{-N} \right)^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} \right. \\
&\quad \left. \sum_{\epsilon', |j-j'| \leq 1, k'} |a_{j', k'}^{\epsilon'}|^p (1 + |2^{j-j'} k' - k|)^{-N} (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Because  $|j - j'| \leq 1$  and  $0 < t < 2^{-2j\beta}$ , we have

$$\int_0^{2^{-2j\beta}} (t2^{2j'\beta})^{m'} \frac{dt}{t} \lesssim 1$$

and so

$$\begin{aligned}
II_{p,q,Q_r,m'}^{\gamma_1, \gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{|w| \leq 2^n} \frac{1}{(1 + |w|)^{N'}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^{\epsilon'}|^p \right]^{\frac{q}{p}} \\
&\quad + |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{jq(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{|w| > 2^n} \frac{1}{(1 + |w|)^{N'}} \sum_{(\epsilon', k') \in S_r^{w, j'}} \frac{|a_{j', k'}^{\epsilon'}|^p}{(2^{nj'} |Q|)^{N'}} \right]^{\frac{q}{p}}.
\end{aligned}$$

Case 4.1:  $q \leq p$ . For this case, by the  $\alpha$ -triangle inequality, we have

$$\begin{aligned} II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q|^{\frac{q\gamma_2}{n}-\frac{q}{p}}}{(1+|w|)^{\frac{qN}{p}}} \sum_{j' \geq -\log_2 r-1} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\ &\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

Case 4.2:  $q > p$ . Using Hölder's inequality, we have

$$\begin{aligned} II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q|^{\frac{q\gamma_2}{n}-\frac{q}{p}}}{(1+|w|)^N} \sum_{j' \geq -\log_2 r-1} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon',k') \in S_r^{w,j'}} |a_{j',k'}^{\epsilon'}|^p \right)^{\frac{q}{p}} \\ &\lesssim \|f\|_{\dot{B}_{p,q}^{\gamma_1,\gamma_2}}. \end{aligned}$$

This completes the proof of (i).

(ii) We prove

$$f \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} \Rightarrow \pi_\phi(f) = \sum_{(\epsilon,j,k) \in \Lambda_n} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon \in \dot{B}_{p,q}^{\gamma_1,\gamma_2}(\mathbb{R}^n).$$

For the dyadic cube  $Q_r$  and  $j \geq -\log_2 r$ , Hölder's inequality and (3.7) imply

$$\begin{aligned} \sum_{(\epsilon,k) \in S_r^j} |a_{j,k}^\epsilon|^p &\lesssim \sum_{(\epsilon,k) \in S_r^j} \left| \sum_{\epsilon', |j-j'| \leq 1, k'} \int_0^\infty \{(\max\{t2^{2j'\beta}, (t2^{2j'\beta})^{-1}\})^{-N} \right. \\ &\quad \times |a_{j',k'}^{\epsilon'}(t)|(1+|2^{j-j'}k'-k|)^{-N}\} \frac{dt}{t} \Big|^p \\ &\lesssim \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} (1+|2^{j-j'}k'-k|)^{-N} \\ &\quad \left( \int_0^\infty (\max\{t2^{2j'\beta}, (t2^{2j'\beta})^{-1}\})^{-N} |a_{j',k'}^{\epsilon'}(t)| \frac{dt}{t} \right)^p \\ &\lesssim \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left( \int_{r2^\beta}^\infty (t2^{2j'\beta})^{-N} |a_{j',k'}^{\epsilon'}(t)| \frac{dt}{t} \right)^p \\ &\quad + \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left( \int_{-2^{j'\beta}}^{r^{2\beta}} (t2^{2j'\beta})^{-N} |a_{j',k'}^{\epsilon'}(t)| \frac{dt}{t} \right)^p \\ &\quad + \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left( \int_0^{2^{-2j'\beta}} (t2^{2j'\beta})^N |a_{j',k'}^{\epsilon'}(t)| \frac{dt}{t} \right)^p \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Now we estimate the terms

$$M_i = |Q|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_i)^{\frac{q}{p}}, \quad i = 1, 2, 3$$

separately. For simplicity, we may assume  $\|f\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}} = 1$ .

First of all, we control  $M_1$ . Because  $f(\cdot, \cdot) \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} \subset \mathbb{B}_{p,q,m,m'}^{\frac{m}{p},\infty}$ , if  $t2^{2j'\beta} > 1$  then

$$|a_{j',k'}^{\epsilon'}(t)| \lesssim (t2^{2j'\beta})^{-\frac{m}{p}} 2^{-\frac{n_{j'}}{2}} 2^{-j'(\gamma_1-\gamma_2)} \leq 2^{-\frac{n_{j'}}{2}} 2^{-j'(\gamma_1-\gamma_2)}.$$

If  $t2^{2j'\beta} < 1$ , then we still have

$$|a_{j',k'}^{\epsilon'}(t)| \leq 2^{-\frac{n_{j'}}{2}} 2^{-j'(\gamma_1-\gamma_2)}.$$

By Hölder's inequality, we get

$$\begin{aligned}
I_1 &\lesssim \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'|\leq 1} \sum_{\epsilon',k'} (1+|2^{j'-j}k'-k|)^{-N} \left( \int_{r2^\beta}^\infty (t2^{2j'\beta})^{-N} |a_{j',k'}^{\epsilon'}(t)| \frac{dt}{t} \right)^p \\
&\lesssim \sum_{|j-j'|\leq 1} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{(\epsilon,k) \in S_r^j} \sum_{(\epsilon',k') \in S_{jk}^{w,j'}} (1+|2^{j'-j}k'-k|)^{-N'} \\
&\quad \times \left( \int_{r2^\beta}^\infty 2^{-\frac{nj'}{2}} 2^{-j'(\gamma_1-\gamma_2)} (t2^{2j'\beta})^{-N} \frac{dt}{t} \right)^p \\
&\lesssim \sum_{|j-j'|\leq 1} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} [2^{-\frac{nj'}{2}} 2^{-j'(\gamma_1-\gamma_2)} (r2^{j'})^{\frac{n}{p}-2\beta N}]^p \\
&\lesssim [2^{-\frac{nj}{2}} 2^{-j(\gamma_1-\gamma_2)} (r2^j)^{\frac{n}{p}-2\beta N}]^p.
\end{aligned}$$

As a consequence, we get that for  $N$  large enough,

$$\begin{aligned}
M_1^{\frac{1}{q}} &= |Q_r|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_1)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\lesssim |Q_r|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-\frac{qnj}{2}} 2^{-qj(\gamma_1-\gamma_2)} (r2^j)^{qn-2q\beta N} \right)^{\frac{1}{q}} \\
&\lesssim 1.
\end{aligned}$$

Next, we estimate  $M_2$ . Because  $|j-j'| \leq 3$ , it is possible to have  $j' = j, j \pm 1, j \pm 2, j \pm 3$ . However, these cases ensure

$$(I_2)^{\frac{q}{p}} \lesssim \left[ \sum_{(\epsilon,k) \in S_r^j} \sum_{\epsilon', |j-j'|\leq 3, k'} \frac{1}{(1+|2^{j-j'}k'-k|)^{N'}} \left| \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)| (t2^{2j'\beta})^N \frac{dt}{t} \right|^p \right]^{\frac{q}{p}}.$$

Also, Hölder's inequality is applied to imply

$$\begin{aligned}
&\left| \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)| (t2^{2j'\beta})^N \frac{dt}{t} \right|^p \\
&\leq \left( \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right) \left( \int_0^{2^{-2j'\beta}} (t2^{2j'\beta})^N \frac{dt}{t} \right)^{p-1} \\
&\lesssim \left( \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(I_2)^{\frac{q}{p}} &\lesssim \left( \sum_{(\epsilon,k) \in S_r^j} \sum_{\epsilon', |j-j'|\leq 3, k'} \frac{1}{(1+|2^{j-j'}k'-k|)^{N'}} \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right)^{\frac{q}{p}} \\
&\lesssim \left( \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'|\leq 3} \sum_{(\epsilon',k') \in S_{jk}^{w,j'}} \frac{1}{(1+|w|)^{N'}} \frac{1}{(2^{nj'})^{N'}} \int_0^{2^{-2j'\beta}} |a_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right)^{\frac{q}{p}}.
\end{aligned}$$

Case 5.1:  $q \leq p$ . Because  $0 < t < 2^{-2j'\beta}$ , one has

$$(t2^{2j'\beta})^N < (t2^{2j'\beta})^{m'} \quad \text{for } N >> m'.$$

Consequently,

$$\begin{aligned} I_2 &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{(2^{nj'}|Q|)^{-\frac{qN'}{p}}}{(1+|w|)^{\frac{qN'}{p}}} \left( \sum_{(\epsilon, k) \in S_r^j} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right)^{\frac{q}{p}} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{(2^{nj'}|Q|)^{-\frac{qN'}{p}}}{(1+|w|)^{\frac{qN'}{p}}} \left( \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right)^{\frac{q}{p}}. \end{aligned}$$

Because  $j \geq -\log_2 r$  and  $j \sim j'$ , it follows that  $(2^{nj'}|Q|)^{-N} < 1$  and

$$|Q|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_2)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \|f\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}.$$

Case 5.2:  $q > p$ . The Hölder inequality implies

$$\begin{aligned} &\left( \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)| (t2^{2j'\beta})^N \frac{dt}{t} \right)^p \\ &\lesssim \left( \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^N \frac{dt}{t} \right) \left( \int_0^{2^{-2j'\beta}} (t2^{2j'\beta})^N \frac{dt}{t} \right)^{p-1} \\ &\lesssim \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}|^p (t2^{2j'\beta})^N \frac{dt}{t} \\ &\lesssim \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}|^p (t2^{2j'\beta})^{m'} \frac{dt}{t}. \end{aligned}$$

When  $j \geq -\log_2 r$  and  $|j - j'| \leq 3$ , we have  $(2^{nj'}|Q|)^{-N} \lesssim 1$ , and then use Hölder's inequality to imply

$$\begin{aligned} &(I_2)^{\frac{q}{p}} \\ &\lesssim \left( \sum_{(\epsilon, k) \in S_r^j} \sum_{|j-j'| \leq 3} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^{N'}} \frac{1}{(2^{nj'}|Q|)^{N'}} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right)^{\frac{q}{p}} \\ &\lesssim \left[ \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \left( \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right)^{\frac{q}{p}} \right] \\ &\quad \times \left( \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN'}{q-p}} \right)^{\frac{q-p}{p}}, \end{aligned}$$

whence getting

$$\begin{aligned} &|Q|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} (I_2)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &\lesssim |Q|^{\frac{\gamma_2}{n}-\frac{1}{p}} \left[ \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-2j'\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\ &\lesssim \|f\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

Finally, we estimate  $M_3$ . By Hölder's inequality, we can see

$$\begin{aligned}
(I_3)^{\frac{q}{p}} &\lesssim \left( \sum_{(\epsilon, k) \in S_r^j} \sum_{|j-j'| \leq 3} \left| \sum_{\epsilon', k'} \frac{1}{(1+|2^{j-j'} k' - k|)^N} \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)| (t2^{2j'\beta})^{-N} \frac{dt}{t} \right|^p \right)^{\frac{q}{p}} \\
&\lesssim \left( \sum_{(\epsilon, k) \in S_r^j} \sum_{|j-j'| \leq 3} \left| \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \frac{1}{(2^{nj'}|Q|)^N} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)| \frac{1}{(t2^{2j'\beta})^N} \frac{dt}{t} \right|^p \right)^{\frac{q}{p}} \\
&\lesssim \left[ \sum_{(\epsilon, k) \in S_r^j} \sum_{|j-j'| \leq 3} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)| (t2^{2j'\beta})^{-N} \frac{dt}{t} \right)^p \right]^{\frac{q}{p}}.
\end{aligned}$$

Applying Hölder's inequality for  $t$  implies

$$\begin{aligned}
&\left( \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)| \frac{1}{(t2^{2j'\beta})^N} \frac{dt}{t} \right)^p \\
&\lesssim \left( \int_{2^{-2j'\beta}}^{r^{2\beta}} \frac{1}{(t2^{2j'\beta})^N} \frac{dt}{t} \right)^{p-1} \left( \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)|^p \frac{1}{(t2^{2j'\beta})^N} \frac{dt}{t} \right) \\
&\lesssim \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m (t2^{2j'\beta})^{m-N} \frac{dt}{t} \\
&\lesssim \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t},
\end{aligned}$$

where we have used the inequality  $(t2^{2j'\beta})^{m-N} \leq 1$ .

Case 6.1:  $q \leq p$ . For this case, we have, by the  $\alpha$ -triangle inequality,

$$(I_3)^{\frac{q}{p}} \lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN'}{p}} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}$$

and

$$|Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (I_3)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \|f\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}.$$

Case 6.2:  $q > p$ . For this case, we have, by Hölder's inequality,

$$(I_3)^{\frac{q}{p}} \lesssim \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^{N'}} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_{2^{-2j'\beta}}^{r^{2\beta}} |a_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}$$

and

$$|Q|^{\frac{\gamma_2}{n} - \frac{1}{p}} \left( \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (I_3)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \|f\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}.$$

Therefore, the proof of (ii) is complete.  $\square$

We close this section by showing the following continuity of the Riesz transforms acting on the Besov-Q spaces; see also [2] and [37] for some related results.

**Theorem 3.9.** *For  $1 < p, q < \infty$ ,  $\gamma_1, \gamma_2 \in \mathbb{R}$ ,  $m > p$ , and  $m' > 0$ , the Riesz transforms  $R_1, R_2, \dots, R_n$  are continuous on  $\mathbf{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ .*

*Proof.* For the sake of convenience, we choose the classical Meyer wavelet basis  $\{\Phi_{j,k}^\epsilon(x)\}_{(\epsilon,j,k) \in \Lambda_n}$ . For any  $g(\cdot, \cdot) \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2}$  and  $l = 1, 2, \dots, n$ , we need to prove  $(R_l g)(\cdot, \cdot) \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2}$ . Write  $g(t, x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x)$ . Then

$$R_l g(t, x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^\epsilon(t) R_l \Phi_{j,k}^\epsilon(x) =: \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x),$$

where  $b_{j,k}^\epsilon(t)$  is defined by

$$\begin{aligned} b_{j,k}^\epsilon(t) &= \langle R_l g(t, \cdot), \Phi_{j,k}^\epsilon \rangle \\ &= \left\langle \sum_{(\epsilon',j',k') \in \Lambda_n} g_{j',k'}^{\epsilon'}(t) R_l \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \right\rangle \\ &= \sum_{(\epsilon',j',k') \in \Lambda_n} g_{j',k'}^{\epsilon'}(t) \langle R_l \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &= \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} g_{j',k'}^{\epsilon'}(t) \langle R_l \Phi_{j',k'}^{\epsilon'}, \Phi_{j,k}^\epsilon \rangle \\ &=: \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}(t). \end{aligned}$$

Because  $R_l$  is a Calderón-Zygmund operator, we get, by (2.4),

$$|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \lesssim 2^{-|j-j'|(\frac{n}{2}+N_0)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N_0}.$$

We divide the argument into four steps.

**Step 1:**  $(R_l g)(\cdot, \cdot) \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,l}$ ,  $l = 1, 2, \dots, n$ . We have

$$\begin{aligned} I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \times \left( \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^\epsilon(t)|^p (t2^{2j\beta})^m \right)^{\frac{q}{p}} \\ &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \left[ \sum_{(\epsilon,k) \in S_r^j} \left| \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}(t) \right|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \left[ \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \left| \sum_{\epsilon',k'} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}(t) \right|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}. \end{aligned}$$

Because  $|j - j'| \leq 1$ , we get

$$|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \lesssim (1 + |k - k'|)^{-(n+N_0)}.$$

For a dyadic cube  $Q_r$ , we denote by  $\widetilde{Q}_r$  the dyadic cube containing  $Q_r$  and has the volume  $2^{8n}|Q_r|$ . Let  $w \in \mathbb{Z}^n$ . And, we always write  $Q_r^w$  for the dyadic

cube  $2^8rw + \widetilde{Q}_r$ . Applying Hölder's inequality to  $k'$ , we have the following estimate:

$$\begin{aligned}
 (3.8) \quad & \sum_{(\epsilon,k) \in S_r^j} \left| \sum_{\epsilon',k'} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}(t) \right|^p \\
 & \lesssim \sum_{(\epsilon,k) \in S_r^j} \left( \sum_{\epsilon',k'} (1 + |k - k'|)^{-(n+N_0)} |g_{j',k'}^{\epsilon'}(t)| \right)^p \\
 & \lesssim \sum_{(\epsilon,k) \in S_r^j} \sum_{\epsilon',k'} (1 + |k - k'|)^{-(n+N_0)} |g_{j',k'}^{\epsilon'}(t)|^p \\
 & \lesssim \sum_{(\epsilon,k) \in S_r^j} \sum_{w \in \mathbb{Z}^n} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} (1 + |k - k'|)^{-(n+N_0)} |g_{j',k'}^{\epsilon'}(t)|^p \\
 & \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon,k) \in S_r^j} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} (1 + |k - k'|)^{-\frac{N_0}{2}} |g_{j',k'}^{\epsilon'}(t)|^p \\
 & \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p.
 \end{aligned}$$

The above estimate yields

$$\begin{aligned}
 I_{p,q,Q_r,m}^{\gamma_1,\gamma_2}(t) & \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 & \quad \left[ \sum_{|j-j'| \leq 1} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
 & \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
 & \quad \sum_{|j-j'| \leq 1} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}.
 \end{aligned}$$

Now we deal with the term

$$\left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}.$$

If  $q \leq p$ , applying the  $\alpha$ -triangle inequality to  $w$ , we obtain that

$$\begin{aligned}
 & \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n+\frac{N_0}{2})} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
 & \leq \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-q(n+\frac{N_0}{2})/p} \left[ \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}.
 \end{aligned}$$



If  $q > p$ , applying Hölder's inequality to  $w$ , we obtain

$$\begin{aligned} & \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n + \frac{N_0}{2})} \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}} \\ & \leq \left( \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n + \frac{N_0}{2})} \right)^{\frac{q-p}{p}} \\ & \quad \times \left\{ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n + \frac{N_0}{2})} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}} \right\}. \end{aligned}$$

The above two estimates imply that there exists a constant  $N'$  such that for any  $p$  and  $q$ ,

$$\begin{aligned} (3.9) \quad & \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n + \frac{N_0}{2})} \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}} \\ & \leq \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}}. \end{aligned}$$

Because  $|j - j'| \leq 1$ , it is obvious that  $j' = j - 1, j, j + 1$ . Using this we get

$$\begin{aligned} & I_{p, q, Q_r, m}^{\gamma_1, \gamma_2}(t) \\ & \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \\ & \quad \left\{ \sum_{(j' - 1) \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j'\beta})^m \right]^{\frac{q}{p}} \right. \\ & \quad + \sum_{j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j'\beta})^m \right]^{\frac{q}{p}} \\ & \quad \left. + \sum_{(j' + 1) \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j'\beta})^m \right]^{\frac{q}{p}} \right\} \\ & =: M_1 + M_2 + M_3. \end{aligned}$$

Because  $j' \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}$ , we have  $j' > -\log_2(2^8 r)$ . Hence, it is easy to see that  $j' \geq \max\{-\log_2(2^8 r), -\frac{\log_2 t}{2\beta}\}$ , and so that

$$\begin{aligned} M_1 + M_2 & \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2(2^8 r), -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ & \quad \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t 2^{2j'\beta})^m \right]^{\frac{q}{p}} \\ & \lesssim \|g\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2}}. \end{aligned}$$

Also, because  $Q_{j', k'} \subset Q_r^w$ , one has  $j' > -\log_2(2^8 r)$ . On the other hand, note that

$$j' + 1 > -\frac{\log_2 t}{2\beta} \implies j' \geq -\frac{\log_2 t}{2\beta} \text{ or } (-\frac{\log_2 t}{2\beta} - 1) < j' < -\frac{\log_2 t}{2\beta}.$$

Thus, one finds

$$\begin{aligned}
M_3 &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq \max\{-\log_2(2^8 r), -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \times \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \right]^{\frac{q}{p}} \\
&\quad + \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2(2^8 r) < j' < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \times \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \right]^{\frac{q}{p}} \\
&=: M_{3,1} + M_{3,2}.
\end{aligned}$$

On the one hand, it is easy to see  $M_{3,1} \lesssim \|g\|_{\mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, l}}$ . On the other hand, because  $j' < -\frac{\log_2 t}{2\beta}$  yields  $t2^{2j'\beta} \leq 1$  and  $(t2^{2j'\beta})^m < 1$  for  $m > 0$ , one gets

$$\begin{aligned}
M_{3,2} &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \\
&\quad \sum_{-\log_2(2^8 r) < j' < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \\
&\lesssim \|g\|_{\mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}}.
\end{aligned}$$

**Step 2:**  $(R_l g)(\cdot, \cdot) \in \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ ,  $l = 1, 2, \dots, n$ . For this, let the radius  $r$  of  $Q_r$  be  $2^{-j_0}$ .

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1, \gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j_0 < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} \left| \sum_{|j-j'| \leq 1} \sum_{\epsilon', k'} g_{j', k'}^{\epsilon'} a_{j, k, j', k'}^{\epsilon, \epsilon'} \right|^p \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j_0 < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{|j-j'| \leq 1} \left[ \sum_{(\epsilon, k) \in S_r^j} \left( \sum_{\epsilon', k'} |g_{j', k'}^{\epsilon'}| |a_{j, k, j', k'}^{\epsilon, \epsilon'}| \right)^p \right]^{\frac{q}{p}}.
\end{aligned}$$

Because  $|j - j'| \leq 1$ , we get  $|a_{j, k, j', k'}^{\epsilon, \epsilon'}| \lesssim (1 + |k - k'|)^{-(n+N_0)}$ . Then, similar to Step 1, we can obtain that there exist  $N' > 0$  such that

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1, \gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j_0 < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \sum_{|j-j'| \leq 1} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-(n + \frac{N_0}{2})} \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - 1} \sum_{j_0 < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \sum_{|j-j'| \leq 1} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}},
\end{aligned}$$

where we have used the estimates (3.8) and (3.9). Because  $|j - j'| \leq 1$ , we have

$$\begin{aligned}
I_{p,q,Q_r}^{\gamma_1,\gamma_2}(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
&\quad \left\{ \sum_{j_0 < j' - 1 < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \right. \\
&\quad + \sum_{j_0 < j' < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \\
&\quad \left. + \sum_{j_0 < j' + 1 < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \right\} \\
&=: M_4 + M_5 + M_6.
\end{aligned}$$

Because  $Q_{j',k'} \subset Q_r^w$ , we always have  $j' \geq j_0 - 8$ , whence finding

$$\begin{aligned}
M_5 + M_6 &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \\
&\quad \sum_{j_0 - 8 < j' < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \\
&\lesssim \|g\|_{\mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}}.
\end{aligned}$$

For  $M_4$ , we have

$$\begin{aligned}
M_4 &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \\
&\quad \sum_{j_0 - 8 < j' < -\frac{\log_2 t}{2\beta}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \\
&\quad + \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \\
&\quad \sum_{j' \geq \max\{j_0 - 8, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p \right]^{\frac{q}{p}} \\
&=: M_{4,1} + M_{4,2}.
\end{aligned}$$

By definition, we know  $M_{4,1} \lesssim \|g\|_{\mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}}$ . For  $M_{4,2}$ , because  $j' > -\frac{\log_2 t}{2\beta}$  ensures  $(t2^{2j'\beta})^m \geq 1$  for  $m > 0$ , we have

$$\begin{aligned}
M_{4,2} &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \\
&\quad \sum_{j' \geq \max\{j_0 - 8, -\frac{\log_2 t}{2\beta}\}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim \|g\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}}.
\end{aligned}$$

**Step 3:**  $(R_l g)(\cdot, \cdot) \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}$ ,  $l = 1, 2, \dots, n$ . We have the following estimate:

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} \left| \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} g_{j',k'}^{\epsilon'}(t) a_{j,k,j',k'}^{\epsilon,\epsilon'} \right|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \left( \sum_{\epsilon',k'} \frac{|g_{j',k'}^{\epsilon'}(t)|}{(1+|k-k'|)^{(n+N_0)}} \right)^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

By the estimates (3.8), we have

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{|j-j'| \leq 1} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-(n+\frac{N_0}{2})} \right. \\
&\quad \left. \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}
\end{aligned}$$

If  $q \leq p$ , applying the  $\alpha$ -triangle inequality to  $w$ , we have

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-q(n+\frac{N_0}{2})/p} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{|j-j'| \leq 1} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}
\end{aligned}$$

If  $q > p$ , we apply Hölder's inequality to get

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-(n+\frac{N_0}{2})} \right)^{\frac{q-p}{p}} \\
&\quad \times \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-(n+\frac{N_0}{2})} \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{|j-j'| \leq 1} \right. \right. \\
&\quad \left. \left. \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \right\}
\end{aligned}$$

Hence we can always obtain that there exists  $N' > 0$  such that for any  $1 < p, q < \infty$ ,

$$\begin{aligned}
(3.10) \quad I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \sum_{|j-j'| \leq 1} \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Because  $j' = j - 1, j, j + 1$ , one gets

$$\begin{aligned}
I_{p,q,Q_r,m}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
&\quad \left\{ \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2(j'-1)\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \right. \\
&\quad + \sum_{j' \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\quad \left. + \sum_{(j'+1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2(j'+1)\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \right\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Because the length of  $Q_r^w$  is  $2^{8-j_0}$ , by  $Q_{j',k'} \subset Q_r^w$  we have  $j' > j_0 - 8$  and thus

$$\begin{aligned}
I_1 + I_2 &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \|g\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}}.
\end{aligned}$$

Meanwhile, for  $I_3$  we have

$$\begin{aligned}
I_3 &\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j'+1)\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \times \left[ \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\quad + |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \times \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&=: I_{3,1} + I_{3,2}.
\end{aligned}$$

By definition, we have  $I_{3,1} \lesssim \|g\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}}$ . For  $I_{3,2}$ , we should notice that if  $0 < t < 2^{-2j'\beta}$  then  $(t2^{2j'\beta})^{m-m'} \leq 1$  for  $m > m'$ . So, we can obtain

$$\begin{aligned}
I_{3,2} &\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} (t2^{2j'\beta})^{m-m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \|g\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

**Step 4:**  $(R_l g)(\cdot, \cdot) \in \mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}$ ,  $l = 1, 2, \dots, n$ . We have the following estimate:

$$\begin{aligned}
II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} &\lesssim |Q_r|^{\frac{p\gamma_2}{n}-1} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \left| \sum_{|j-j'| \leq 1} \sum_{\epsilon',k'} g_{j',k'}^{\epsilon'}(t) a_{j,k,j',k'}^{\epsilon,\epsilon'} \right|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \sum_{|j-j'| \leq 1} \left( \sum_{\epsilon',k'} \frac{|g_{j',k'}^{\epsilon'}(t)|}{(1+|k-k'|)^{(n+N_0)}} \right)^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-(n+\frac{N_0}{2})} \sum_{|j-j'| \leq 1} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}},
\end{aligned}$$

where we have used (3.8) in the last inequality.

If  $q \leq p$ , we apply the  $\alpha$ -triangle inequality to  $w$ . If  $q > p$ , we apply Hölder's inequality to  $w$ . Similar to (3.10), we obtain that there exists  $N' > 0$  such that

$$\begin{aligned}
II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq j_0} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \sum_{|j-j'| \leq 1} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Since  $j' = j - 1, j, j + 1$ , we further get

$$\begin{aligned}
& II_{p,q,Q_r,m'}^{\gamma_1,\gamma_2} \\
& \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \left\{ \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2(j'-1)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \right. \\
& \quad + \sum_{j' \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& \quad \left. + \sum_{(j'+1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2(j'+1)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \right\} \\
& =: I_4 + I_5 + I_6.
\end{aligned}$$

Because  $Q_{j',k'} \subset Q_r^w$  ensures  $j' > j_0 - 8$ , we have

$$\begin{aligned}
I_5 + I_6 & \lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \quad \sum_{j' \geq j_0-8} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& \lesssim \|g\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

For  $I_4$ , we have

$$\begin{aligned}
I_4 & \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \quad \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2(j'-1)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& \lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \quad \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& \quad + |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \quad \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{2^{-2(j'-1)\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& =: I_{4,1} + I_{4,2}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_{4,1} & \lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\
& \quad \sum_{j' \geq (j_0-8)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |g_{j',k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
& \lesssim \|g\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

For  $I_{4,2}$  let  $r_w = 2^{8-j_0}$ . Then  $\log_2 r_w = 8 - j_0$ . Thanks to  $j' - 1 > j_0$  and  $t < 2^{-2(j'-1)\beta}$ , we have

$$t < 2^{-2(j'-1)\beta} < 2^{-2j_0\beta} = 2^{-2(8-\log_2 r_w)\beta} \leq r_w^{2\beta}.$$

Finally, we get, by using  $(t2^{2j'\beta})^{m'-m} \leq 1$ , the following estimate:

$$\begin{aligned} I_{4,2} &\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\ &\quad \sum_{(j'-1) \geq j_0} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{2^{-2(j'-1)\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\ &\lesssim |Q_r^w|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N'} \\ &\quad \sum_{j' \geq -\log_2 r_w} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^{r_w^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |g_{j', k'}^{\epsilon'}(t)|^p (t2^{2j'\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\ &\lesssim \|g\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}}, \end{aligned}$$

thereby completing the proof of Theorem 3.9.  $\square$

#### 4. NON-LINEAR TERMS AND THEIR A PRIOR ESTIMATES

##### 4.1. Decompositions of non-linear terms. From now on, let

$$\begin{cases} u(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} u_{j, k}^{\epsilon}(t) \Phi_{j, k}^{\epsilon}(x); \\ v(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} v_{j, k}^{\epsilon}(t) \Phi_{j, k}^{\epsilon}(x). \end{cases}$$

For  $l = 1, \dots, n$ , we will derive some inequalities about

$$B_l(u, v)(t, x) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l}(uv) ds.$$

Here, it is worth pointing out that (1.1) gives

$$e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l}(uv)(s, t, x) = \sum_{j' \in \mathbb{Z}} \sum_{i=1}^5 I_{j'}^{i, l}(s, t, x),$$



where

$$\begin{aligned}
I_{j'}^{1,l}(u, v)(s, t, x) &= \sum_{\epsilon', k'} \sum_{k''} u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)), \\
I_{j'}^{2,l}(u, v)(s, t, x) &= \sum_{\epsilon', k'} \sum_{\epsilon'', k''} u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)), \\
I_{j'}^{3,l}(u, v)(s, t, x) &= \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)), \\
I_{j'}^{4,l}(u, v)(s, t, x) &= \sum_{0 < j'' - j' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)), \\
I_{j'}^{5,l}(u, v)(s, t, x) &= \sum_{\epsilon', k'} \sum_{k''} v_{j', k'}^{\epsilon'}(s) u_{j'-3, k''}^0(s) \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)).
\end{aligned}$$

Hence

$$\begin{aligned}
B_l(u, v)(t, x) &=: \int_0^t \sum_{j' \in \mathbb{Z}} \sum_{i=1}^5 I_{j'}^{i,l}(s, t, x) ds \\
&= \sum_{i=1}^5 \int_0^t \sum_{j' \in \mathbb{Z}} I_{j'}^{i,l}(s, t, x) ds \\
&=: \sum_{i=1}^5 \int_0^t I_l^i(s, t, x) ds.
\end{aligned}$$

Therefore, we can write

$$(4.1) \quad B_l(u, v)(t, x) := \sum_{i=1}^5 I_l^i(u, v)(t, x),$$

where

$$I_l^i(u, v)(t, x) = \int_0^t I_l^i(s, t, x) ds.$$

In order to estimate the bilinear term  $B(u, v)$  in some function spaces on  $\mathbb{R}^n$ , we are required to decompose the terms  $I_l^i(u, v)(t, x)$ ,  $i = 1, 2, \dots, 5$ , respectively.

**Decomposition of  $I_l^1(u, v)(t, x)$ .** The term  $I_l^1(u, v)(t, x)$  is decomposed according to two cases.

Case  $[I_l^1]_1$ :  $t \geq 2^{-2j\beta}$ . For this case, we write  $I_l^1(u, v)(t, x)$  as the sum of the following three terms:

$$\begin{aligned}
I_l^{1,1}(u, v)(t, x) &= \sum_{\epsilon', j', k'} \sum_{k''} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) \right. \\
&\quad \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \Big\} ds;
\end{aligned}$$

$$\begin{aligned}
I_l^{1,2}(u, v)(t, x) &= \sum_{\epsilon', j', k'} \sum_{k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds; \\
I_l^{1,3}(u, v)(t, x) &= \sum_{\epsilon', j', k'} \sum_{k''} \int_{\frac{t}{2}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'-3, k''}^0(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds.
\end{aligned}$$

For  $i = 1, 2, 3$ , denote

$$I_l^{1,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Case  $[I_l^1]_2$ :  $t < 2^{-2j\beta}$ . For this case, we denote  $a_{j, k}^{\epsilon, 4}(t) = a_{j, k}^\epsilon(t)$  and then have

$$I_l^1(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x).$$

**Decomposition of  $I_l^2(u, v)(t, x)$ .** The decomposition of  $I_l^2(u, v)(t, x)$  is made according to two cases.

Case  $[I_l^2]_1$ :  $t \geq 2^{-2j\beta}$ . Naturally,  $I_l^2(u, v)(t, x)$  can be divided into the following three terms:

$$\begin{aligned}
I_l^{2,1}(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds; \\
I_l^{2,2}(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds; \\
I_l^{2,3}(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds.
\end{aligned}$$

Case  $[I_l^2]_2$ :  $t \leq 2^{-2j\beta}$ . This  $I_l^2(u, v)(t, x)$  can be decomposed into the sum of  $II^4(u, v)(t, x)$  and  $II^5(u, v)(t, x)$ , where

$$\begin{aligned}
I_l^{2,4}(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds
\end{aligned}$$

and

$$\begin{aligned}
I_l^{2,5}(u, v)(t, x) &= \sum_{j'} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j', k''}^{\epsilon''}(s) \right. \\
&\quad \times \left. e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j', k''}^{\epsilon''}(x)) \right\} ds.
\end{aligned}$$

For  $i = 1, 2, 3, 4, 5$ , set

$$I_l^{2,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

**Decompositions of  $I_l^3(u, v)(t, x)$ .** Similarly, we have the following two cases:

Case  $[I_l^3]_1$ :  $t \geq 2^{-2j\beta}$ . This  $I_l^3(u, v)(t, x)$  can be divided into the following three terms:

$$I_l^{3,1}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds;$$

$$I_l^{3,2}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds;$$

$$I_l^{3,3}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds.$$

Case  $[I_l^3]_2$ :  $t \leq 2^{-2j\beta}$ . This  $I_l^3(u, v)(t, x)$  can be decomposed into the sum of  $II^4(u, v)(t, x)$  and  $II^5(u, v)(t, x)$ , where

$$I_l^{3,4}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds$$

and

$$I_l^{3,5}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds.$$

For  $i = 1, 2, 3, 4, 5$ , denote

$$I_l^{3,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

**Decompositions of  $I_l^4(u, v)(t, x)$ .** These can be done by considering two following cases.

Case  $[I_l^4]_1$ :  $t \geq 2^{-2j\beta}$ . This  $I_l^4(u, v)(t, x)$  can be divided into the following three terms:

$$I_l^{4,1}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds;$$

$$I_l^{4,2}(u, v)(t, x) = \sum_{0 < j' - j'' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds;$$

$$I_l^{4,3}(u, v)(t, x) = \sum_{0 < j'' - j' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds.$$

Case  $[I_l^4]_2$ :  $t \leq 2^{-2j\beta}$ . This  $I_l^4(u, v)(t, x)$  can be decomposed into the sum of  $II^4(u, v)(t, x)$  and  $II^5(u, v)(t, x)$ , where

$$I_l^{4,4}(u, v)(t, x) = \sum_{0 < j'' - j' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-2j'\beta}} \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds$$

and

$$I_l^{4,5}(u, v)(t, x) = \sum_{0 < j'' - j' \leq 3} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ u_{j', k'}^{\epsilon'}(s) v_{j'', k''}^{\epsilon''}(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'', k''}^{\epsilon''}(x)) \right\} ds.$$

For  $i = 1, 2, 3, 4, 5$ , denote

$$I_l^{4,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

**Decomposition of  $I_j^{5,l}(u, v)(t, x)$ .** It is easy to see that the terms  $I_j^{1,l}(u, v)(t, x)$  and  $I_j^{5,l}(u, v)(t, x)$  are symmetric associated with  $u(t, x)$  and  $v(t, x)$ . Hence for  $I_l^5(u, v)$ , we have a similar decomposition.

Case  $[I_l^5]_1$ :  $t \geq 2^{-2j\beta}$ . For this case, we write  $I_l^5(u, v)(t, x)$  as the sum of the following three terms:

$$I_l^{5,1}(u, v)(t, x) = \sum_{\epsilon', j', k'} \sum_{k''} \int_0^{2^{-1-2j'\beta}} \left\{ v_{j', k'}^{\epsilon'}(s) u_{j'-3, k''}^0(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds;$$

$$I_l^{5,2}(u, v)(t, x) = \sum_{\epsilon', j', k'} \sum_{k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ v_{j', k'}^{\epsilon'}(s) u_{j'-3, k''}^0(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds;$$

$$I_l^{5,3}(u, v)(t, x) = \sum_{\epsilon', j', k'} \sum_{k''} \int_{\frac{t}{2}}^t \left\{ v_{j', k'}^{\epsilon'}(s) u_{j'-3, k''}^0(s) \right. \\ \left. \times e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j', k'}^{\epsilon'}(x) \Phi_{j'-3, k''}^0(x)) \right\} ds.$$

For  $i = 1, 2, 3$ , denote

$$I_l^{5,i}(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x).$$

Case  $[I_l^5]_2$ :  $t < 2^{-2j\beta}$ . For this case, we denote  $a_{j, k}^{\epsilon, 4}(t) = a_{j, k}^\epsilon(t)$  and then have

$$I_l^1(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x).$$

**4.2. Induced a prior estimates.** In what follows, we are about to dominate the above-defined  $a_{j,k}^{\epsilon,1}$ ,  $b_{j,k}^{\epsilon,1}$ ,  $u_{j',k'}^{\epsilon'}$  and  $v_{j',k''}^{\epsilon''}$ .

**Lemma 4.1.**

(i)

$$|a_{j,k}^{\epsilon,1}(t)| \lesssim 2^{\frac{n_j}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \right. \\ \left. \times e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k|)^{-N} \right\} ds;$$

(ii)

$$|a_{j,k}^{\epsilon,2}(t)| \lesssim 2^{\frac{n_j}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \right. \\ \left. \times e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k|)^{-N} \right\} ds;$$

(iii)

$$|a_{j,k}^{\epsilon,3}(t)| \lesssim 2^{\frac{n_j}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \\ \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k|)^{-N} ds;$$

(iv)

$$|a_{j,k}^{\epsilon,4}(t)| \lesssim 2^{\frac{n_j}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_0^t |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \\ \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k|)^{-N} ds.$$

*Proof.*

$$a_{j,k}^{\epsilon,1}(t) = \left\langle I_l^{1,1}(u, v), \Phi_{j,k}^\epsilon \right\rangle \\ = \sum_{\epsilon',k',k''} \sum_{|j-j'|\leq 2} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j',k'}^{\epsilon'}(s) v_{j'-3,k''}^0(s) \right. \\ \left. \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right\} ds.$$

Upon writing  $e^{-(t-s)(-\Delta)^\beta} = \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} (-\Delta)^{m\beta}$ , we have

$$\left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \\ = - \left\langle \Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0, e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} \Phi_{j,k}^\epsilon \right\rangle \\ = - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \left\langle \Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0, (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} \Phi_{j,k}^\epsilon \right\rangle \\ = - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \int_{\mathbb{R}^n} \left\{ 2^{\frac{n_j'}{2}} \Phi^{\epsilon'}(2^{j'}x - k') 2^{\frac{n(j'-3)}{2}} \Phi^0(2^{j'-3}x - k'') \right. \\ \left. (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} [2^{\frac{n_j}{2}} \Phi^\epsilon(2^jx - k)] \right\} dx.$$

If  $\Phi_1^\epsilon$  stands for the function  $(-\Delta)^{m\beta} \frac{\partial}{\partial x_l} \Phi^\epsilon$ , then

$$\begin{aligned}
& \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \\
&= - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \int_{\mathbb{R}^n} \left\{ 2^{\frac{nj'}{2}} \Phi^{\epsilon'}(2^{j'}x - k') 2^{\frac{n(j'-3)}{2}} \Phi^0(2^{j'-3}x - k) 2^{\frac{nj}{2}} \right. \\
&\quad \left. 2^{(2m\beta+1)j} \Phi_1^\epsilon(2^jx - k) \right\} dx \\
&= - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \int_{\mathbb{R}^n} 2^{nj'} \left\{ 2^{\frac{nj}{2}} 2^{(2m\beta+1)j} \Phi^{\epsilon'}(2^{j'}x - 2^{j'-j}k - k') \right. \\
&\quad \left. \Phi^0(2^{j'-3}x - 2^{j'-j-3}k - k'') \Phi_1^\epsilon(2^jx) \right\} dx \\
&= - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} 2^{nj'} 2^{\frac{nj}{2}} 2^{(2m\beta+1)j} 2^{-nj} \int_{\mathbb{R}^n} \left\{ \Phi^{\epsilon'}(2^{j'-j}x - 2^{j'-j}k - k') \right. \\
&\quad \left. \Phi^0(2^{j'-j-3}x - 2^{j'-j-3}k - k'') \Phi_1^\epsilon(x) \right\} dx.
\end{aligned}$$

For  $\epsilon \neq 0$ , denote by  $i_\epsilon$  the smallest one of the indexes  $i$  with  $\epsilon_i \neq 0$ . For  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , write  $y_0 = x_{i_\epsilon}$ . For any  $N \geq 1$ , denote by  $x_N^\epsilon$  the vector  $(x_1, \dots, x_{i_\epsilon-1}, y_N, x_{i_\epsilon-1}, \dots, x_n)$ . For any  $\Phi(x)$ , write

$$I_N^\epsilon \Phi(x) = \int_{-\infty}^{x_{i_\epsilon}} \cdots \int_{-\infty}^{y_{N-1}} \Phi(x_N^\epsilon) dy_1 \cdots dy_N.$$

Denote by  $D_N^\epsilon \Phi(x)$  the function  $\frac{\partial^N}{\partial x_{i_\epsilon}^N} \Phi(x)$ .

Because  $0 < s < \frac{t}{2}$  gives  $(t-s) \sim t$ , by  $j \sim j'$  we have

$$\begin{aligned}
& \left| \left\langle e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j'-3,k''}^0), \Phi_{j,k}^\epsilon \right\rangle \right| \\
&= \left| - \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} 2^{2m\beta j} 2^{j+\frac{nj}{2}} \int_{\mathbb{R}^n} \left\{ I_N^\epsilon \Phi_1^\epsilon(x) \right. \right. \\
&\quad \left. \left. \times D_N^\epsilon [\Phi^\epsilon(2^{j'-j}x - 2^{j'-j}k - k') \Phi^0(2^{j'-j-3}x - 2^{j'-j-3}k - k'')] \right\} dx \right| \\
&\lesssim e^{-ct2^{2j\beta}} 2^{\frac{nj}{2}+j} \int_{\mathbb{R}^n} \left\{ (1+|x|)^{-N_1} (1+|2^{j'-j}x - 2^{j'-j}k - k'|)^{-N_2} \right. \\
&\quad \left. \times (1+|2^{j'-j-3}x - 2^{j'-j-3}k - k''|)^{-N_2} \right\} dx \\
&\lesssim e^{-ct2^{2j\beta}} 2^{\frac{nj}{2}+j} \left( \int_A + \int_B + \int_C \right) (1+|x|)^{-N_1} (1+|2^{j'-j}x - 2^{j'-j}k - k'|)^{-N_2} \\
&\quad \times (1+|2^{j'-j-3}x - 2^{j'-j-3}k - k''|)^{-N_2} dx \\
&=: e^{-ct2^{2j\beta}} 2^{\frac{nj}{2}+j} (I_1 + I_2 + I_3),
\end{aligned}$$

where

$$\begin{cases} A = \left\{ x \in \mathbb{R}^n : |2^{j'-j}x| > \frac{1}{2}|2^{j'-j}k - k'| \text{ and } |2^{j'-j-3}x| > \frac{1}{2}|2^{j'-j-3}k - k''| \right\}; \\ B = \left\{ x \in \mathbb{R}^n : |2^{j'-j}x| < \frac{1}{2}|2^{j'-j}k - k'| \text{ and } |2^{j'-j-3}x| < \frac{1}{2}|2^{j'-j-3}k - k''| \right\}; \\ C = \mathbb{R}^n \setminus (A \cup B). \end{cases}$$

For  $I_1$ , using  $j \sim j'$  we get

$$\begin{aligned} I_1 &\leq \left( \int_{|x| > 2^{j-j'-1}|2^{j'-j}k-k'|} (1+|x|)^{-N_1} dx \right)^{\frac{1}{2}} \\ &\quad \times \left( \int_{|x| > 2^{j-j'-4}|2^{j'-j-3}k-k''|} (1+|x|)^{-N_2} dx \right)^{\frac{1}{2}} \\ &\leq (1+|2^{j-j'}k-k'|)^{-\frac{N_1}{2}} (1+|2^{j'-j-3}k-k''|)^{-\frac{N_2}{2}}. \end{aligned}$$

Note that

$$x \in B \implies \begin{cases} |2^{j'-j}x - 2^{j'-j}k - k'| \gtrsim |2^{j'-j}k - k'|; \\ |2^{j'-j-3}x - 2^{j'-j-3}k - k''| \gtrsim |2^{j'-j-3}k - k''|. \end{cases}$$

So, for  $I_2$ , using  $j \sim j'$  we obtain

$$\begin{aligned} I_2 &\lesssim (1+|2^{j'-j}k-k'|)^{-N_2} (1+|2^{j'-j-3}k-k''|)^{-N_2} \int_B (1+|x|)^{-N_1} dx \\ &\lesssim (1+|2^{j-j'}k'-k|)^{-N_2} (1+|2^{j-j'+3}k''-k|)^{-N_2}. \end{aligned}$$

For  $I_3$ , we have

$$\begin{aligned} I_3 &\leq (1+|2^{j'-j-3}k-k''|)^{-N_2} \int_{|x| \geq \frac{1}{2}2^{j-j'}|2^{j'-j}k-k'|} (1+|x|)^{-N_1} dx \\ &\quad + (1+|2^{j'-j}k-k'|)^{-N_2} \int_{|x| \geq \frac{1}{2}2^{j-j'+3}|2^{j'-j-3}k-k''|} (1+|x|)^{-N_1} dx \\ &\lesssim (1+|2^{j-j'}k'-k|)^{-N_2} (1+|2^{j-j'+3}k''-k|)^{-N_2}. \end{aligned}$$

This completes the proof of Lemma 4.1 (i). In a similar manner, we can prove Lemma 4.1 (ii)/(iii)/(iv).  $\square$

**Lemma 4.2.**

(i)

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \\ &\quad \times e^{-\tilde{c}t2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} (1+|2^{j-j'}k''-k|)^{-N} \} ds; \end{aligned}$$

(ii)

$$\begin{aligned} |b_{j,k}^{\epsilon,2}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \\ &\quad \times e^{-\tilde{c}t2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} (1+|2^{j-j'}k''-k|)^{-N} \} ds; \end{aligned}$$

(iii)

$$\begin{aligned} |b_{j,k}^{\epsilon,3}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{\frac{t}{2}}^t \{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \\ &\quad \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} (1+|2^{j-j'}k''-k|)^{-N} \} ds; \end{aligned}$$

(iv)

$$\begin{aligned} |b_{j,k}^{\epsilon,4}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_0^{2^{-2j'\beta}} \{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \\ &\quad \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} (1+|2^{j-j'}k''-k|)^{-N} \} ds; \end{aligned}$$

(v)

$$|b_{j,k}^{\epsilon,5}(t)| \lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \right. \\ \left. \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'}k'' - k|)^{-N} \right\} ds.$$

*Proof.* Following the argument for Lemma 4.1, we only prove (i). By definition,

$$b_{j,k}^{\epsilon,1}(t) = \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \sum_{j < j'+2} \int_0^{2^{-1-2j'\beta}} \left\{ u_{j',k'}^{\epsilon'}(s) v_{j',k''}^{\epsilon''}(s) \right. \\ \left. \langle e^{-c(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j',k''}^{\epsilon''}), \Phi_{j,k}^\epsilon \rangle \right\} ds.$$

Formally, we can write

$$\begin{aligned} & \left\langle e^{-c(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j',k''}^{\epsilon''}), \Phi_{j,k}^\epsilon \right\rangle \\ &= \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \left\langle (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j',k''}^{\epsilon''}), \Phi_{j,k}^\epsilon \right\rangle \\ &= - \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} \left\langle (\Phi_{j',k'}^{\epsilon'} \Phi_{j',k''}^{\epsilon''}), (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} (\Phi_{j,k}^\epsilon) \right\rangle. \end{aligned}$$

The inner product in the above equality can be written as

$$\begin{aligned} & \int_{\mathbb{R}^n} \Phi_{j',k'}^{\epsilon'}(x) \Phi_{j',k''}^{\epsilon''}(x) (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} (\Phi_{j,k}^\epsilon(x)) dx \\ &= \int_{\mathbb{R}^n} 2^{\frac{nj'}{2}} \Phi^{\epsilon'}(2^{j'}x - k') 2^{\frac{nj''}{2}} \Phi^{\epsilon''}(2^{j'}x - k'') (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} (2^{\frac{nj}{2}} \Phi^\epsilon(2^jx - k)) dx \\ &= \int_{\mathbb{R}^n} \left\{ 2^{nj'} \Phi^{\epsilon'}(2^{j'}x - k') \Phi^{\epsilon''}(2^{j'}x - k'') 2^{\frac{nj}{2}} 2^{(2m\beta+1)j} \right. \\ & \quad \left. \times ((-\Delta)^{m\beta} \frac{\partial}{\partial x_l} \Phi^\epsilon)(2^jx - k) \right\} dx. \end{aligned}$$

Let  $\Phi_1^\epsilon = (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} \Phi^\epsilon$ . By the change of variables, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \Phi_{j',k'}^{\epsilon'}(x) \Phi_{j',k''}^{\epsilon''}(x) (-\Delta)^{m\beta} \frac{\partial}{\partial x_l} (\Phi_{j,k}^\epsilon(x)) dx \\ &= 2^{nj'} 2^{\frac{nj}{2}} 2^{j(2m\beta+1)} \int_{\mathbb{R}^n} \Phi^{\epsilon'}(2^{j'}x) \Phi^{\epsilon''}(2^jx + k' - k'') \Phi_1^\epsilon(2^jx + 2^{j-j'}k' - k) dx \\ &= 2^{\frac{nj}{2}+j} 2^{2m\beta j} \int_{\mathbb{R}^n} \Phi^{\epsilon'}(y) \Phi^{\epsilon''}(2^{j-j'}y + k' - k'') \Phi_1^\epsilon(2^{j-j'}y + 2^{j-j'}k' - k) dy \\ &= 2^{\frac{nj}{2}+j} 2^{2m\beta j} \int_{\mathbb{R}^n} \left\{ I_N^\epsilon \Phi^{\epsilon'}(y) D_N^\epsilon [\Phi^{\epsilon''}(2^{j-j'}y + k' - k'') \right. \\ & \quad \left. \times \Phi_1^\epsilon(2^{j-j'}y + 2^{j-j'}k' - k)] \right\} dy, \end{aligned}$$



where  $I_N^\epsilon$  and  $D_N^\epsilon$  are defined in the proof of Lemma 4.1. Finally, by  $t - s \sim t$  and  $j - j' < 2$ , we can get

$$\begin{aligned}
& \left| \left\langle e^{-c(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (\Phi_{j',k'}^{\epsilon'} \Phi_{j',k''}^{\epsilon''}), \Phi_{j,k}^\epsilon \right\rangle \right| \\
&= \left| \sum_{m=0}^{\infty} \frac{[-(t-s)]^m}{m!} 2^{\frac{mj}{2}+j} 2^{2m\beta j} \int_{\mathbb{R}^n} \left\{ I_N^\epsilon \Phi^{\epsilon'}(y) \right. \right. \\
&\quad \left. \left. \times D_N^\epsilon \left[ \Phi^{\epsilon''}(2^{j-j'}y + k' - k'') \Phi_1^\epsilon(2^{j-j'}y + 2^{j-j'}k' - k) \right] \right\} dy \right| \\
&= \left| 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \int_{\mathbb{R}^n} I_N^\epsilon \Phi^{\epsilon'}(y) D_N^\epsilon \left[ \Phi^{\epsilon''}(2^{j-j'}y + k' - k'') \right. \right. \\
&\quad \left. \left. \times \Phi_1^\epsilon(2^{j-j'}y + 2^{j-j'}k' - k) \right] dy \right| \\
&\leq 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} 2^{(j-j')N} \int_{\mathbb{R}^n} \left\{ (1 + |y|)^{-N} (1 + |2^{j-j'}y + k' - k''|)^{-N} \right. \\
&\quad \left. \times (1 + |2^{j-j'}y + 2^{j-j'}k' - k|)^{-N} \right\} dy \\
&= 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} 2^{(j-j')N} \left( \int_D + \int_E + \int_F \right) \left\{ (1 + |y|)^{-N} (1 + |2^{j-j'}y + k' - k''|)^{-N} \right. \\
&\quad \left. \times (1 + |2^{j-j'}y + 2^{j-j'}k' - k|)^{-N} \right\} dy,
\end{aligned}$$

where

$$\begin{cases} D = \{y \in \mathbb{R}^n : |2^{j-j'}y| > \frac{1}{2}|2^{j-j'}k' - k| \text{ and } |2^{j-j'}y| > \frac{1}{2}|k' - k''|\}; \\ E = \{y \in \mathbb{R}^n : |2^{j-j'}y| > \frac{1}{2}|2^{j-j'}k' - k| \text{ and } |2^{j-j'}y| > \frac{1}{2}|k' - k''|\}; \\ F = \mathbb{R}^n \setminus (D \cup E). \end{cases}$$

In the same way as proving Lemma 4.1, we can reach the desired estimate (i). The estimates (ii)-(v) can be proved similarly.  $\square$

In the forthcoming lemmas, let  $Q_{j,k}$  and  $Q_{j',k'}$  be two dyadic cubes, and for  $w \in \mathbb{Z}^n$  denote by  $Q_{j,k}^w$  the dyadic cube  $\widetilde{Q}_{j,k} + 2^{8-j}w$ , where  $\widetilde{Q}_{j,k}$  denotes the dyadic cube containing  $Q_{j,k}$  with side length  $2^{8-j}$ .

**Lemma 4.3.** *For  $j, j' \in \mathbb{Z}$  and  $w, k, k' \in \mathbb{Z}^n$ , if  $Q_{j',k'} \subset Q_{j,k}^w$ , then*

$$(4.2) \quad (1 + |2^{j-j'}k' - k|)^{-N} \lesssim (1 + |w|)^{-N}.$$

*Proof.* If  $x_0$  is the center of  $Q_{j,k}$ , then an application of the triangle inequality implies

$$\begin{aligned}
|2^{j-j'}k' - k| &= |2^j|2^{-j'}k' - 2^{-j}k| \\
&\geq 2^j(2^{8-j}|w| - |2^{-j'}k' - 2^{8-j}w - x_0| - |x_0 - 2^{-j}k|) \\
&\geq 2^j(2^{8-j}|w| - 2^{8-j} - 2^{-j}).
\end{aligned}$$

Hence, for  $|w| \geq 2(2^8 + 1) > 1$ , we have

$$(1 + |2^{j-j'}k' - k|)^{-N} \lesssim [1 + (|w| - 2^8 - 1)]^{-N} \lesssim (|w|)^{-N} \lesssim (1 + |w|)^{-N}.$$

If  $|w| < 2(2^8 + 1)$ , then the above inequality is obvious, and hence (4.2) follows.  $\square$

**Lemma 4.4.** *Let  $0 < j' - j'' \leq 3$ ,  $j \leq j' + 5$  and  $|w - w'| > 2^n$ . If  $Q_{j',k'} \subset Q_{j,k}^w$  and  $Q_{j'',k''} \subset Q_{j,k}^{w'}$ , then*

$$(4.3) \quad (1 + |2^{j'-j''}k'' - k'|)^{-N} \lesssim 2^{N(j-j')}(1 + |w - w'|)^{-N}.$$

*Proof.* Using the assumption, we estimate

$$\begin{aligned} |2^{j'-j''}k'' - k'| &\geq 2^{j'} \left( |2^{8-j}w - 2^{8-j}w'| - |2^{-j''}k'' - 2^{8-j}w' - x_0| \right. \\ &\quad \left. - |2^{-j'}k' - 2^{8-j}w - x_0| \right) \\ &\geq 2^{j'}(2^{8-j}|w - w'| - 2^{8-j} - 2^{8-j}) \\ &\gtrsim 2^{j'-j}(|w - w'| - 2) \\ &\gtrsim 2^{j'-j}|w - w'|, \end{aligned}$$

whence getting

$$(1 + |2^{j'-j''}k'' - k'|)^{-N} \lesssim 2^{(j-j')N}(|w - w'|)^{-N}.$$

By  $|w - w'| > 2^n$ , we can get (4.3).  $\square$

**Lemma 4.5.** *Let  $Q_{j,k}$  be a dyadic cube with radius  $2^{-j}$ . For  $w \in \mathbb{Z}^n$ , set  $Q_{j,k}^w$  be the dyadic cube  $2^{8-j}w + \tilde{Q}_{j,k}$ . Then*

$$\begin{aligned} (4.4) \quad &\sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j',k'}^{\epsilon'}(s)| |v_{j'',k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'}k' - k|)^{-8N} (1 + |k' - k''|)^{-8N} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} \sum_{w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \\ &\quad \times \left( \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w',j''}} |v_{j'',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}. \end{aligned}$$

*Proof.* If  $Q_{j',k'} \subset Q_{j,k}^w$ , then  $(1 + |2^{j-j'}k' - k|)^{-N} \lesssim (1 + |w|)^{-N}$ . If  $Q_{j',k'} \subset Q_{j,k}^w$  and  $Q_{j'',k''} \subset Q_{j,k}^{w'}$ , then  $(1 + |k' - k''|)^{-N} \leq (1 + |w - w'|)^{-N}$ , and hence

$$\begin{aligned} &(1 + |2^{j-j'}k' - k|)^{-2N} (1 + |k' - k''|)^{-2N} \\ &\leq (1 + |w|)^{-2N} (1 + |w - w'|)^{-2N} \\ &\lesssim (1 + |w|)^{-N} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \\ &\lesssim (1 + |w|)^{-N} (1 + |w - w'| + |w|)^{-N} \\ &\lesssim (1 + |w|)^{-N} (1 + |w'|)^{-N}. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
& \sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'} k' - k|)^{-8N} (1 + |k' - k''|)^{-8N} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \sum_{w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \\
& \quad \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |k' - k''|)^{-6N} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \sum_{w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \\
& \quad \times \left[ \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |k' - k''|)^{-6N} \right)^{p'} \right]^{\frac{1}{p'}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \left[ \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |k' - k''|)^{-6N} \right)^{p'} \right]^{\frac{1}{p'}} \\
& \leq \left[ \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (1 + |k' - k''|)^{-6N} \right)^{\frac{p'}{p}} \right]^{\frac{1}{p'}} \\
& \leq \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
& \sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'} k' - k|)^{-8N} (1 + |k' - k''|)^{-8N} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \sum_{w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \\
& \quad \times \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}},
\end{aligned}$$

thereby reaching (4.4).  $\square$

**Lemma 4.6.** *Let  $Q_{j,k}$  be a dyadic cube with radius  $2^{-j}$ . For  $w \in \mathbb{Z}^n$ , denote by  $Q_{j,k}^w$  the dyadic cube  $2^{8-j}w + \widetilde{Q}_{j,k}$ . If  $\delta > 0$  is small enough, then*

$$\begin{aligned}
(4.5) \quad & \sum_{(\epsilon, k) \in S_r^j} \left\{ \sum_{j \leq j' + 5} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |a_{j', k'}^{\epsilon}|^p \right)^{\frac{1}{p}} \right\}^p \\
& \lesssim \sum_{j \leq j' + 5} 2^{\delta(j' - j)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_r^{w, j}} |a_{j', k'}^{\epsilon}|^p.
\end{aligned}$$

*Proof.* Applying Hölder's inequality to  $k'$  and  $j'$ , respectively, we have

$$\begin{aligned}
& \sum_{(\epsilon, k) \in S_r^j} \left\{ \sum_{j \leq j' + 5} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |a_{j', k'}^\epsilon|^p \right)^{\frac{1}{p}} \right\}^p \\
& \lesssim \sum_{j \leq j' + 5} 2^{\delta(j' - j)} \sum_{(\epsilon, k) \in S_r^j} \left\{ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |a_{j', k'}^\epsilon|^p \right\}^{\frac{1}{p}} \\
& \lesssim \sum_{j \leq j' + 5} 2^{\delta(j' - j)} \sum_{(\epsilon, k) \in S_r^j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |a_{j', k'}^\epsilon|^p.
\end{aligned}$$

Changing the order of summation, we get

$$\begin{aligned}
& \sum_{(\epsilon, k) \in S_r^j} \left\{ \sum_{j \leq j' + 5} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |a_{j', k'}^\epsilon|^p \right)^{\frac{1}{p}} \right\}^p \\
& \lesssim \sum_{j \leq j' + 5} 2^{\delta(j' - j)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_r^{w, j'}} |a_{j', k'}^\epsilon|^p,
\end{aligned}$$

whence getting (4.5).  $\square$

*Remark 4.7.* In the proof of Lemma 4.6, we have used the following fact. For fixed  $j$ , the number of  $Q_{j', k'}$  which are contained in the dyadic cube  $Q_{j, k}^w = 2^{8-j}w + \tilde{Q}_{j, k}$  equals to  $2^{n(8+j'-j)}$ . On the other hand, for any dyadic cube  $Q_r$  with radius  $r$ , the number of  $Q_{j, k} \subset Q_r$  equals to  $(2^j r)^n$ . Then the number of  $Q_{j', k'}$  which are contained in the dyadic cube  $Q_r^w$  equals  $(2^{8+j'} r)^n$ . In the proof of the main lemmas in Sections 7-11, we will use this fact again.

**Lemma 4.8.** *Let  $Q_{j, k}$  be a dyadic cube with radius  $2^{-j}$ . For  $w \in \mathbb{Z}^n$ , denote by  $Q_{j, k}^w$  the dyadic cube  $2^{8-j}w + \tilde{Q}_{j, k}$ . If  $j < j' + 2$ , then*

$$\begin{aligned}
& \sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| (1 + |2^{j'-j}k' - k|)^{-N} (1 + |k' - k''|)^{-N} \\
(4.6) \quad & \lesssim \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} 2^{n(j' - j)(1 - \frac{2}{p})} \\
& \quad \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

*Proof.* Applying Hölder's inequality to  $k$  and  $k'$  respectively, we obtain, by (4.3),

$$\begin{aligned}
& \sum_{\epsilon', k'} \sum_{\epsilon'', k''} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \\
& \lesssim \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)| (1 + |2^{j-j'} k' - k|)^{-N} \\
& \quad \times \left[ \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (1 + |k' - k''|)^{-N} \right]^{\frac{1}{p}} \\
& \lesssim \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \left( \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} (1 + |2^{j-j'} k' - k|)^{-N} \right)^{1 - \frac{2}{p}} \\
& \quad \times \left( \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left[ \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} \frac{|v_{j', k''}^{\epsilon''}(s)|^p}{(1 + |k' - k''|)^N} \right]^{\frac{1}{p}} \\
& \lesssim \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} 2^{n(j-j)(1-\frac{2}{p})} \\
& \quad \times \left( \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}},
\end{aligned}$$

whence reaching (4.6).  $\square$

## 5. PROOF OF THE MAIN THEOREM

By Picard's contraction principle and Theorems 3.8 & 3.9, it is enough to verify that the bilinear operator

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P} \nabla \cdot (u \otimes v) ds$$

is bounded from  $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n \times (\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$  to  $(\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2})^n$ . To do so, let

$$B_l(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_l} (uv) ds$$

and

$$B_{l,l'}(u, v) = R_l R_{l'} \int_0^t e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_{l'}} (uv) ds.$$

We need to prove that all  $B_l(u, v)$ ,  $B_{l,l'}(u, v)$  are bounded from  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2} \times \mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$  to  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ . Because  $R_{l'}$ ,  $l' = 1, \dots, n$  are bounded on  $\mathbb{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ , we only consider the boundedness of  $B_l(u, v)$ . By (4.1), if

$$u(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} u_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x) \quad \text{and} \quad v(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} v_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x),$$

then

$$B_l(u, v)(t, x) = \sum_{i=1}^5 I_l^i(u, v)(t, x),$$

where the terms  $I_l^i(u, v)(t, x)$ ,  $i = 1, 2, \dots, 5$  are defined in Subsection 4.1.

It is easy to see that the argument for  $I_l^5(u, v)(t, x)$  is similar to that for  $I_l^1(u, v)(t, x)$ . Also the treatments of  $I_l^3(u, v)(t, x)$  and  $I_l^4(u, v)(t, x)$  are similar

to that of  $I_l^2(u, v)(t, x)$ . So, we are only required to show that the following functions

$$(t, x) \mapsto I_l^1(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^\epsilon(t) \Phi_{j, k}^\epsilon(x)$$

and

$$(t, x) \mapsto I_l^2(u, v)(t, x) = \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^\epsilon(t) \Phi_{j, k}^\epsilon(x)$$

belong to  $\mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ . By the decompositions of non-linear terms obtained in Subsection 4.1, it is equivalent to prove that the following functions

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \phi_{j, k}^\epsilon(x), \quad i = 1, 2, 3, 4$$

and

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \phi_{j, k}^\epsilon(x), \quad i = 1, 2, 3, 4, 5,$$

are members of  $\mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ . The demonstration will be concluded by proving the following five lemmas under the conditions of Theorem 0.1:

**Lemma 5.1.** *If  $(\beta, p, q, \gamma_1, \gamma_2, m, m')$  satisfies the conditions of Theorem 0.1 and  $u, v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ , then*

- (i) *For  $i = 1, 2, 3$ , the function  $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x)$  is in  $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, I}$ ;*
- (ii) *For  $i = 1, 2, 3$ , the function  $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x)$  is in  $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$ ;*
- (iii) *The function  $(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x)$  is in  $\mathbb{B}_{p, q}^{\gamma_1, \gamma_2, II} \cap \mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}$ .*

**Lemma 5.2.** *If  $(\beta, p, q, \gamma_1, \gamma_2, m, m')$  satisfies the conditions of Theorem 0.1 and  $u, v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ , then the functions*

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x), \quad i = 1, 2, 3$$

*are in  $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, I}$ .*

**Lemma 5.3.** *If  $(\beta, p, q, \gamma_1, \gamma_2, m, m')$  satisfies the conditions of Theorem 0.1 and  $u, v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ , then the functions*

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, i}(t) \Phi_{j, k}^\epsilon(x), \quad i = 1, 2, 3$$

*are in  $\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}$ .*

**Lemma 5.4.** *If  $(\beta, p, q, \gamma_1, \gamma_2, m, m')$  satisfies the conditions of Theorem 0.1 and  $u, v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}$ , then the functions*

$$\begin{cases} (t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^\epsilon(x); \\ (t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 5}(t) \Phi_{j, k}^\epsilon(x), \end{cases}$$

are in  $\mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ .

**Lemma 5.5.** *If  $(\beta, p, q, \gamma_1, \gamma_2, m, m')$  satisfies the conditions of Theorem 0.1 and  $u, v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ , then the functions*

$$\begin{cases} (t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x); \\ (t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon,5}(t) \Phi_{j,k}^\epsilon(x), \end{cases}$$

are in  $\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}$ .

## 6. PROOF OF LEMMA 5.1

**6.1. The setting (i).** According to the relation between  $2^{-2j\beta}$  and  $t$ , we divide the proof into three cases.

Case 6.1:

$$(t, x) \mapsto \sum_{(\epsilon, j, k)} a_{j,k}^{\epsilon,1}(t) \Phi_{j,k}^\epsilon(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}.$$

For simplicity, we assume

$$\|u\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}} = \|v\|_{\mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}} = 1.$$

Because  $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ , one has  $v \in \mathbb{B}_{\frac{m}{p},\infty}^{\gamma_1-\gamma_2} \subset \mathbb{B}_{0,\infty}^{\gamma_1-\gamma_2}$ . Hence

$$|v_{j'-3,k''}^0(s)| \lesssim s^{-\frac{\gamma_2-\gamma_1}{2\beta}} 2^{-\frac{nj'}{2}},$$

and consequently, by (i) of Lemma 4.1,

$$\begin{aligned} |a_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \right. \\ &\quad \left. \times e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'+3}k'' - k|)^{-N} \right\} ds \\ &\lesssim 2^j \sum_{|j-j'|\leq 2} \sum_{\epsilon',k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| (1 + |2^{j-j'}k' - k|)^{-N} e^{-\tilde{c}t2^{2j\beta}} s^{\frac{1}{2\beta}} \frac{ds}{s}. \end{aligned}$$

Notice that  $|j - j'| \leq 2$ . So, applying Hölder's inequality to  $k'$  we get

$$\begin{aligned}
& I_{Q_r}^1(t) \\
&= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{pj} \sum_{|j-j'| \leq 2} \right. \\
&\quad \left. \left( \sum_{\epsilon', k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}} \frac{ds}{s} \right)^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} 2^{pj} e^{-\tilde{c}pt2^{2j\beta}} \right. \\
&\quad \left. (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}.
\end{aligned}$$

By  $p > 2m'\beta$ ,  $j \sim j'$  and Hölder's inequality, we have

$$\begin{aligned}
& \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \\
&\lesssim \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p s^{m' \frac{ds}{s}} \right) \left( \int_0^{2^{-1-2j'\beta}} s^{(\frac{1}{2\beta} - \frac{m'}{p})p' \frac{ds}{s}} \right)^{p-1} \\
&\lesssim 2^{-pj} \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} ds \right).
\end{aligned}$$

Hence by the inequality (4.2), we get

$$\begin{aligned}
& I_{Q_r}^1(t) \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{pj} e^{-cpt2^{2j\beta}} \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-pj} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} (t2^{2j\beta})^m \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 6.1.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we obtain

$$\begin{aligned}
I_{Q_r}^1(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \\
&\quad \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}} \lesssim 1.
\end{aligned}$$

Subcase 6.1.2:  $q > p$ . Hölder's inequality implies that

$$\begin{aligned}
I_{Q_r}^1(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \\
&\quad \left[ \sum_{(\epsilon', k') \in S_r^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}} \lesssim 1.
\end{aligned}$$



Case 6.2:

$$(t, x) \mapsto \sum_{(\epsilon, j, k)} a_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, I}.$$

Thanks to

$$\begin{cases} v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}; \\ \gamma_1 - \gamma_2 = 1 - 2\beta, \end{cases}$$

we have

$$|v_{j'-3, k'}^0(s)| \lesssim s^{\frac{1}{2\beta}-1} 2^{-\frac{n_{j'}}{2}}.$$

and consequently, by (ii) of Lemma 4.1,

$$\begin{aligned} |a_{j, k}^{\epsilon, 2}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon', k', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j'-3, k'}^0(s)| \right. \\ &\quad \left. \times e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'}+3k'' - k|)^{-N} \right\} ds \\ &\lesssim 2^j \sum_{|j-j'|\leq 2} \sum_{\epsilon', k'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds. \end{aligned}$$

Hence, by the above estimate of  $|a_{j, k}^{\epsilon, 2}(t)|$  and  $|j - j'| \leq 2$ , we get

$$\begin{aligned} I_Q^2(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |a_{j, k}^{\epsilon, 2}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{jp} e^{-\tilde{c}pt2^{2j\beta}} \right. \\ &\quad \left. \sum_{|j-j'|\leq 2} \left( \sum_{\epsilon', k'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \right]^{\frac{q}{p}}. \end{aligned}$$

Write

$$A_{j, k} = \left( \sum_{\epsilon', k'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds \right)^p.$$

Firstly, we assume  $t > r^{2\beta}$ . By Hölder's inequality, we have

$$\begin{aligned} A_{j, k} &\lesssim \sum_{\epsilon', k'} (1 + |2^{j-j'}k' - k|)^{-N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \\ &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^m ds \right) \\ &\quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{(\frac{1}{2\beta}-1-\frac{m}{p})p'} ds \right)^{\frac{p}{p'}} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-j'(p-2m\beta-2\beta)} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^m ds \right). \end{aligned}$$

The above estimate yields

$$I_{Q_r}^2(t) \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ \sum_{|j-j'| \leq 2} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{pj} 2^{-pj'} 2^{2\beta j'} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \right. \\ \left. \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m ds \right) \right]^{\frac{q}{p}}.$$

Because  $u \in \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, I}$ , one has

$$\sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \lesssim 2^{p\gamma_2 j - n j} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}.$$

Thus

$$I_{Q_r}^2(t) \lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^{N'}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ \sum_{|j-j'| \leq 2} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{p(j-j')} 2^{2\beta j'} 2^{p\gamma_2 j - n j} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} t (t2^{2j\beta})^m e^{-\tilde{c}t2^{2j\beta}} \right]^{\frac{q}{p}} \\ \lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1 + |w|)^{N'}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} (t2^{2j\beta})^{\frac{q(1+m)}{p}} e^{-\tilde{c}t2^{2j\beta}} [2^{p\gamma_2 j - n j} (2^j r)^n]^{\frac{q}{p}} \\ \leq \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n}}}{(1 + |w|)^{N'}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} (t2^{2j\beta})^{\frac{q(1+m)}{p} + \frac{q\gamma_2}{2\beta}} t^{-\frac{q\gamma_2}{2\beta}} e^{-\tilde{c}t2^{2j\beta}} \\ \lesssim 1,$$

where we have used the assumption  $t > r^{2\beta}$ .

Secondly, we assume  $t \leq r^{2\beta}$ . A combination of Hölder's inequality and (4.2) implies

$$A_{j,k} \lesssim \sum_{\epsilon', k'} (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \\ \lesssim \sum_{\epsilon', k'} \frac{1}{(1 + |2^{j-j'} k' - k|)^N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \right) \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{(\frac{1}{2\beta} - \frac{m}{p})p'} \frac{ds}{s} \right)^{\frac{1}{p'}} \\ \lesssim 2^{-2j'\beta(\frac{1}{2\beta} - \frac{m}{p})p} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \\ \lesssim 2^{-j'p} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s}.$$

This in turn gives

$$I_Q^2(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}}.$$

Subcase 6.2.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned} I_Q^2(t) &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \times \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_{2^{-1-2j'\beta}}^{\frac{1}{2}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\ &\lesssim \|u\|_{\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}} \lesssim 1. \end{aligned}$$

Subcase 6.2.2:  $q > p$ . Hölder's inequality implies that

$$\begin{aligned} I_Q^2(t) &\lesssim \sum_{w \in \mathbb{Z}^n} |Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \times \left[ \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_{2^{-1-2j'\beta}}^{\frac{1}{2}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\ &\lesssim \|u\|_{\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}} \lesssim 1. \end{aligned}$$

Case 6.3:

$$(t, x) \mapsto \sum_{(\epsilon, j, k)} a_{j, k}^{\epsilon, 3}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, I}.$$

Because  $v \in \mathbb{B}_{p, q, m, m'}^{\gamma_1, \gamma_2} \subset \mathbb{B}_{0, \infty}^{\gamma_1, \gamma_2}$ , we similarly have, by (iii) of Lemma 4.1,

$$\begin{aligned} |a_{j, k}^{\epsilon, 3}(t)| &\lesssim 2^{\frac{nj}{2} + j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_{\frac{1}{2}}^t \left\{ |u_{j', k'}^{\epsilon'}(s)| v_{j'-3, k''}^0(s) \right. \\ &\quad \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |2^{j-j'} k'' - k|)^{-N} \Big\} ds \\ &\lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_{\frac{1}{2}}^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds. \end{aligned}$$

The above estimate for  $|a_{j, k}^{\epsilon, 3}(t)|$  implies

$$\begin{aligned} I_Q^3(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k) \in S_r^j} |a_{j, k}^{\epsilon, 3}|^p (t 2^{2j\beta})^m \right)^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} 2^{jp} \sum_{|j-j'| \leq 2} \right. \\ &\quad \left. \left( \sum_{\epsilon', k'} \int_{\frac{1}{2}}^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds \right)^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}}. \end{aligned}$$

Applying Hölder's inequality on  $k'$  and  $s$  respectively, we can get

$$\begin{aligned} &\left( \sum_{\epsilon', k'} \int_{\frac{1}{2}}^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds \right)^p \\ &\lesssim \sum_{\epsilon', k'} (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{\frac{1}{2}}^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} s^{\frac{1}{2\beta}-1} ds \right)^p \\ &\lesssim \sum_{\epsilon', k'} (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{\frac{1}{2}}^t |u_{j', k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} s^m \frac{ds}{s} \right) \\ &\quad \times \left( \int_{\frac{1}{2}}^t e^{-\tilde{c}(t-s)2^{2j\beta}} s^{(\frac{1}{2\beta} - \frac{m}{p})p'} \frac{ds}{s} \right)^{\frac{p}{p'}}, \end{aligned}$$

where we have used  $|j - j'| \leq 2$ .

On the other hand, it is easy to derive

$$\begin{aligned} 2^{pj} \left( \int_{\frac{t}{2}}^t e^{-\tilde{c}(t-s)2^{2j\beta}} s^{\frac{(p-2\beta m)}{2\beta(p-1)}} \frac{ds}{s} \right)^{p-1} &\lesssim 2^{pj} \left( t^{\frac{(p-2\beta m)}{2\beta(p-1)}} \right)^{p-1} \left( \int_{\frac{t}{2}}^t e^{-\tilde{c}(t-s)2^{2j\beta}} ds \right)^{p-1} \\ &\lesssim 2^{pj} t^{\frac{(p-2\beta m)}{2\beta} - (p-1)} 2^{-2j\beta(p-1)} \\ &= (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1)} t^{-m}, \end{aligned}$$

where  $(2\beta - 1)p > 2\beta(1 - m)$ . Hence by the inequality (4.2), we can obtain

$$\begin{aligned} I_Q^3(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{|j-j'| \leq 2} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\ &\quad \left. \int_{\frac{t}{2}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} s^m \frac{ds}{s} (t2^{2j\beta})^m (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1)} t^{-m} \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{|j-j'| \leq 2} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\ &\quad \left. \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} (s2^{2j'\beta})^m \frac{ds}{s} \right) (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1)} \right]^{\frac{q}{p}}. \end{aligned}$$

Because  $u \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ , it follows that

$$\sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \lesssim r^{n-p\gamma_2} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}.$$

Since  $\beta > \frac{1}{2}$ , one has  $\frac{p}{2\beta} < p$ . Consequently, we get

$$\begin{aligned} I_Q^3(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{|j-j'| \leq 2} \\ &\quad \left[ \left( \int_{\frac{t}{2}}^t e^{-\tilde{c}(t-s)2^{2j\beta}} r^{n-p\gamma_2} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \frac{ds}{s} \right) (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1)} \right]^{\frac{q}{p}} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left[ (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1) - 1} \right]^{\frac{q}{p}} \\ &\quad \times \left( \int_0^t e^{-\tilde{c}u2^{2j\beta}} d(u2^{2j\beta}) \right)^{\frac{q}{p}} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N'} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} (t2^{2j\beta})^{q(\frac{p}{2\beta} - p)/p} \\ &\lesssim 1. \end{aligned}$$

**6.2. The setting (ii).** To prove that for each  $i = 1, 2, 3$  the function

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, i}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III},$$

we consider three cases.

Case 6.4:

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j,k}^{\epsilon, 1}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}.$$

By  $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2}$ , we get, by (i) of Lemma 4.1,

$$\begin{aligned} |a_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{n}{2}+j} \sum_{|j-j'|\leq 2} \sum_{\epsilon',k',k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \right. \\ &\quad \left. \times e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j'}k'' - k|)^{-N} ds \right\} \\ &\lesssim 2^j \sum_{|j-j'|\leq 2} \sum_{\epsilon',k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}} \frac{ds}{s}, \end{aligned}$$

whence obtaining

$$\begin{aligned} I_{Q_r}^1 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |a_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{pj} e^{-\tilde{c}pt2^{2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \right. \\ &\quad \left. \left( \sum_{|j-j'|\leq 2} \sum_{\epsilon',k'} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| (1 + |2^{j-j'}k' - k|)^{-N} s^{\frac{1}{2\beta}} \frac{ds}{s} \right)^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}. \end{aligned}$$

Applying Hölder's inequality to  $k'$  and  $s$  respectively, as well as (4.2) and  $|j-j'|\leq 2$ , we find

$$\begin{aligned} &\left( \sum_{\epsilon',k'} (1 + |2^{j-j'}k' - k|)^{-N} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \\ &\leq \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \\ &\leq \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p s^{m'} \frac{ds}{s} \right) \left( \int_0^{2^{-1-2j'\beta}} s^{(\frac{1}{2\beta}-\frac{m'}{p})p'} \frac{ds}{s} \right)^{\frac{p}{p'}} \\ &\leq 2^{j(2\beta m'-p)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \left( \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p s^{m'} \frac{ds}{s} \right) \\ &= 2^{-jp} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right), \end{aligned}$$

whence reaching

$$\begin{aligned} I_{Q_r}^1 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{q}{p})} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{jp} e^{-\tilde{c}pt2^{2j\beta}} \right. \\ &\quad \left. \sum_{(\epsilon,k) \in S_r^j} 2^{-jp} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} \int_0^{2^{-1-2j'\beta}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}. \end{aligned}$$

The rest of the proof is divided into two subcases.

Subcase 6.4.1:  $q \leq p$ . By changing variables, we find

$$\int_{2^{-2j\beta}}^{r^{2\beta}} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \frac{dt}{t} \leq \int_1^\infty e^{-cu} u^m \frac{du}{u} \lesssim 1,$$

thereby getting

$$\begin{aligned}
I_{Q_r}^1 &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \\
&\quad \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \left( \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-1-2j'\beta}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Subcase: 6.4.2:  $q > p$ . Applying Hölder's inequality for  $w$  and  $\frac{q}{p} > 1$ , we similarly have

$$\begin{aligned}
I_{Q_r}^1 &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \\
&\quad \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-\tilde{c}t2^{2j\beta}} (t2^{2j\beta})^m \left( \sum_{(\epsilon', k') \in S_r^{w, j'}} \int_0^{2^{-1-2j'\beta}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^{m'} \frac{ds}{s} \right) \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 6.5:

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}.$$

For this, by (ii) of Lemma 4.1, we have

$$\begin{aligned}
|a_{j, k}^{\epsilon, 2}(t)| &\lesssim 2^{\frac{n}{2}j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left\{ |u_{j', k'}^{\epsilon'}(s)| v_{j'-3, k''}^0(s) \right. \\
&\quad \times \left. e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |2^{j-j'} k'' - k|)^{-N} \right\} ds \\
&\lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}t2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds.
\end{aligned}$$

This last estimate, along with the Hölder inequality on  $k'$ , implies

$$\begin{aligned}
I_{Q_r}^2 &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} |a_{j, k}^{\epsilon, 2}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} 2^{pj} e^{-cpt2^{2j\beta}} \right. \\
&\quad \left. \left( \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \frac{|u_{j', k'}^{\epsilon'}(s)|}{(1 + |2^{j-j'} k' - k|)^N} s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{q}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} 2^{pj} e^{-cpt2^{2j\beta}} \right. \\
&\quad \left. \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Using Hölder's inequality once again, we have

$$\begin{aligned} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j',k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p &\leq \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j',k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \right) \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{\frac{p-2\beta m}{2\beta(p-1)}} \frac{ds}{s} \right)^{p-1} \\ &\lesssim 2^{-j'p} 2^{2j'\beta m} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j',k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \right), \end{aligned}$$

whence producing

$$\begin{aligned} I_{Q_r}^2 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} 2^{pj} e^{-cpt^{2j\beta}} \sum_{|j-j'| \leq 2} 2^{-j'p} 2^{2j'\beta m} \right. \\ &\quad \left. \sum_{\epsilon',k'} (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} |u_{j',k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \right) (t^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \end{aligned}$$

Subcase: 6.5.1:  $q \leq p$ . Because  $|j - j'| \leq 2$ , we change the order of integration to get

$$\begin{aligned} I_{Q_r}^2 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}}, \end{aligned}$$

where we have used (4.2) and

$$\int_{2^{-1-2j'\beta}}^{\frac{t}{2}} e^{-\tilde{c}pt^{2j\beta}} (t^{2j\beta})^m \frac{dt}{t} \lesssim 1.$$

Subcase: 6.5.2:  $q > p$ . Similarly, Hölder's inequality and the inequality (4.2) imply

$$\begin{aligned} I_{Q_r}^2 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ &\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\ &\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}}. \end{aligned}$$

Case 6.6:

$$(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} a_{j,k}^{\epsilon,3}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}.$$

Similarly, we have, by (iii) of Lemma 4.1,

$$\begin{aligned} |a_{j,k}^{\epsilon,3}(t)| &\lesssim 2^{\frac{n}{2}+j} \sum_{|j-j'| \leq 2} \sum_{\epsilon',k',k''} \int_{\frac{t}{2}}^t \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j'-3,k''}^0(s)| \right. \\ &\quad \left. \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |2^{j-j'+3} k'' - k|)^{-N} \right\} ds \\ &\lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon',k'} \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds. \end{aligned}$$

Hence we can get, by Hölder's inequality and  $|j - j'| \leq 2$ ,

$$\begin{aligned}
I_{Q_r}^3 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} |a_{j,k}^{\epsilon,3}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} 2^{pj} \left[ \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)| \right. \right. \\
&\quad \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta}-1} ds \Big]^p (t2^{2j\beta})^m \frac{dt}{t} \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} 2^{pj} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \right. \\
&\quad \left. (1 + |2^{j-j'} k' - k|)^{-N} \left( \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} s^{\frac{1}{2\beta}-1} ds \right)^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Applying Hölder's inequality to  $s$  yields

$$\begin{aligned}
&\left( \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} s^{\frac{1}{2\beta}-1} ds \right)^p \\
&\leq t^{(\frac{1}{2\beta}-1)p} \left( \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} ds \right) \left( \int_{\frac{t}{2}}^t e^{-\tilde{c}(t-s)2^{2j\beta}} ds \right)^{p-1} \\
&\leq t^{(\frac{1}{2\beta}-1)p} 2^{-2j\beta(p-1)} \left( \int_{\frac{t}{2}}^t |u_{j',k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} ds \right).
\end{aligned}$$

By the above estimate and (4.2), we deduce

$$\begin{aligned}
I_{Q_r}^3 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r - 2} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\
&\quad \left. \int_{2^{-(j+2)\beta}}^{r^{2\beta}} 2^{pj-2j\beta(p-1)} t^{\frac{p}{2\beta}-p} \int_{\frac{t}{2}}^t \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p e^{-\tilde{c}(t-s)2^{2j\beta}} ds (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 6.6.1:  $q \leq p$ . Changing the order of integration, we have

$$\begin{aligned}
I_{Q_r}^3 &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r - 2} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \left( \int_s^{2s} e^{-\tilde{c}(t-s)2^{2j\beta}} (t2^{2j\beta})^{\frac{p}{2\beta}-(p-1)} \frac{dt}{t} \right) (s2^{2j\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}}.
\end{aligned}$$

From  $2^{-2j\beta} \leq s \leq r^{2\beta}$  and  $\beta > \frac{1}{2}$  it follows readily that

$$\begin{aligned}
\int_s^{2s} e^{-\tilde{c}(t-s)2^{2j\beta}} (t2^{2j\beta})^{\frac{p}{2\beta}-(p-1)} \frac{dt}{t} &\leq (s2^{2j\beta})^{\frac{p}{2\beta}-(p-1)} \int_s^{2s} e^{-\tilde{c}(t-s)2^{2j\beta}} \frac{dt}{t} \\
&\leq (s2^{2j\beta})^{\frac{p}{2\beta}-(p-1)} \int_{s2^{2j\beta}}^{2s2^{2j\beta}} e^{-\tilde{c}u \frac{du}{u}} e^{cs2^{2j\beta}} \\
&\leq (s2^{2j\beta})^{\frac{p}{2\beta}-p} \lesssim 1.
\end{aligned}$$

Finally we have

$$\begin{aligned}
I_{Q_r}^3 &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j' \geq -\log_2 r - 2} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j+5)\beta-1}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$



Subcase 6.6.2:  $q > p$ . Applying Hölder's inequality to  $w$ , we have

$$\begin{aligned}
I_{Q_r}^3 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-\frac{qN}{p}} \sum_{j' \geq -\log_2 r - 2} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \left( \int_s^{2s} e^{-\tilde{c}(t-s)2^{2j\beta}} (t2^{2j\beta})^{\frac{p}{2\beta} - (p-1)\frac{dt}{t}} (s2^{2j\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right] \\
&\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-\frac{qN}{p}} \sum_{j' \geq -\log_2 r - 2} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$

**6.3. The setting (iii).** The proof of that the function

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q}^{\gamma_1, \gamma_2, II} \cap \mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}$$

is divided into two parts.

Case 6.7:

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q}^{\gamma_1, \gamma_2, II}.$$

For this case we have, by (iv) of Lemma 4.1,

$$\begin{aligned}
|a_{j, k}^{\epsilon, 4}(t)| &\lesssim 2^{\frac{n}{2} + j} \sum_{|j-j'| \leq 2} \sum_{\epsilon', k', k''} \int_0^t \{ |u_{j', k'}^{\epsilon'}(s)| |v_{j'-3, k''}^0(s)| \\
&\quad \times e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |2^{j-j'} k' - k|)^{-N} \} ds \\
&\lesssim 2^j \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta} - 1} ds.
\end{aligned}$$

The above estimate and (4.2) imply that

$$\begin{aligned}
I_{Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k) \in S_r^j} |a_{j, k}^{\epsilon, 4}(t)|^p \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} 2^{pj} \left[ \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^t |u_{j', k'}^{\epsilon'}(s)| e^{-\tilde{c}(t-s)2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} s^{\frac{1}{2\beta} - 1} ds \right]^p \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ 2^{pj} \sum_{|j-j'| \leq 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\
&\quad \left. \sum_{(\epsilon', k') \in S_r^{w, j'}} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta} - 1} ds \right)^p \right\}^{\frac{q}{p}}.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
\left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta} - 1} ds \right)^p &\leq \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta} - 1 + \mu)p} ds \right) \left( \int_0^t s^{-\mu p'} ds \right)^{\frac{p}{p'}} \\
&\leq t^{p-1-\mu p} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta} - 1 + \mu)p} ds \right).
\end{aligned}$$

So

$$I_{Q_r}^4(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ 2^{pj} \sum_{|j-j'|\leq 2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\ \left. \sum_{(\epsilon', k') \in S_r^{w, j'}} t^{p-1-\mu p} \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right]^{\frac{q}{p}}$$

Subcase 6.7.1:  $q > p$ . By Hölder's inequality, we have

$$I_{Q_r}^4(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \\ 2^{qj} \sum_{|j-j'|\leq 2} t^{\frac{(p-1-p\mu)q}{p}} \left[ \int_0^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right]^{\frac{q}{p}}.$$

Applying Hölder's inequality again, we have

$$\left( \int_0^t \left( \sum_{\epsilon', k'} |u_{j', k'}^{\epsilon'}(s)|^p \right) s^{(\frac{1}{2\beta}-1+\mu)p} ds \right)^{\frac{q}{p}} \\ \leq \left[ \int_0^t \left( \sum_{\epsilon', k'} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{q}{p}} s^{(\frac{1}{2\beta}-1+\mu)p} ds \right] \left[ \int_0^t s^{(\frac{1}{2\beta}-1+\mu)p} ds \right]^{\frac{q-p}{p}} \\ \leq t^{(\frac{p}{2\beta}-p+p\mu+1)(\frac{q}{p}-1)} \left[ \int_0^t \left( \sum_{\epsilon', k'} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{q}{p}} s^{(\frac{1}{2\beta}-1+\mu)p} ds \right].$$

Because  $2^{2j\beta}t \leq 1$ , one has  $2^{qj} \leq t^{-\frac{q}{2\beta}}$ . This in turn gives

$$I_{Q_r}^4(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{qj} \\ \sum_{|j-j'|\leq 2} t^{\frac{q}{2\beta}} t^{-p(\frac{1}{2\beta}-1+\mu)-1} \left[ \int_0^t \left( \sum_{\epsilon', k'} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{q}{p}} s^{(\frac{1}{2\beta}-1+\mu)p} ds \right] \\ \lesssim \|u\|_{\mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}} 2^{qj} t^{\frac{q}{2\beta}} t^{-p(\frac{1}{2\beta}-1+\mu)-1} t^{(\frac{1}{2\beta}-1+\mu)p+1} \\ \lesssim \|u\|_{\mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}}.$$

Subcase 6.7.2:  $q \leq p$ . We have

$$I_{Q_r}^4(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\ \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{|j-j'|\leq 2} 2^{jp} \sum_{(\epsilon', k') \in S_r^{w, j'}} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \right]^{\frac{q}{p}}.$$

Because  $t \leq 2^{-2j\beta}$  and  $0 < m' < \min\{1, \frac{p}{2\beta}\}$ , Hölder's inequality implies

$$\left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p \leq \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{m' \frac{ds}{s}} \right) \left( \int_0^t s^{(\frac{1}{2\beta}-\frac{m'}{p})p' \frac{ds}{s}} \right)^{\frac{p}{p'}} \\ \lesssim 2^{-j(p-2\beta m')} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{m' \frac{ds}{s}} \right).$$

Finally, we obtain

$$\begin{aligned}
& I_{Q_r}^4(t) \\
& \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \\
& \quad \sum_{|j-j'| \leq 2} \left[ 2^{pj} \int_0^{2^{-2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^{m'} 2^{-2j'\beta(\frac{p}{2\beta}-m')} \frac{ds}{s} \right]^{\frac{q}{p}} \\
& \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
& \lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 6.8:

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} a_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}.$$

Similarly, we can get

$$\begin{aligned}
I_{Q_r}^4 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} |a_{j, k}^{\epsilon, 4}(t)|^p (t 2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} 2^{pj} \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{|j-j'| \leq 2} \sum_{\epsilon', k'} \int_0^t |u_{j', k'}^{\epsilon'}(s)| \right. \right. \\
&\quad \left. \left. e^{-\tilde{c}(t-s)2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} s^{\frac{1}{2\beta}-1} ds \right]^p (t 2^{2j\beta})^{m'} \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} 2^{pj} \sum_{|j-j'| \leq 2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \left. \sum_{(\epsilon', k') \in S_r^{w, j'}} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p (t 2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}},
\end{aligned}$$

where we have used (4.2) and Hölder's inequality on  $k'$  and  $s$ .

Choosing a constant  $\mu$  such that  $m' + p - 1 - \frac{p}{2\beta} \leq p\mu < p - 1$ , we use Hölder's inequality to get

$$\begin{aligned}
\left( \int_0^t |u_{j', k'}^{\epsilon'}(s)| s^{\frac{1}{2\beta}-1} ds \right)^p &\leq \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right) \left( \int_0^t s^{-\mu p'} ds \right)^{\frac{p}{p'}} \\
&\lesssim t^{p-1-\mu p} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right),
\end{aligned}$$

whence reaching

$$\begin{aligned}
I_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} 2^{pj} \sum_{|j-j'| \leq 2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \left. \sum_{(\epsilon', k') \in S_r^{w, j'}} t^{p-1-\mu p} \left( \int_0^t |u_{j', k'}^{\epsilon'}(s)|^p s^{(\frac{1}{2\beta}-1+\mu)p} ds \right) (t 2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 6.8.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned}
I_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{|j-j'|\leq 2} 2^{qj} \\
&\quad \left[ \int_0^{2^{-2j\beta}} \int_0^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^{\frac{1}{2\beta}-1+\mu)p} ds (t2^{2j\beta})^{m'} t^{p-1-\mu p} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \sum_{|j-j'|\leq 2} 2^{qj} \\
&\quad \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p s^{\frac{p}{2\beta}-p+p\mu} \left( \int_s^{2^{-2j\beta}} 2^{2mj'\beta} t^{m'+p-\mu p} \frac{dt}{t^2} \right) ds \right]^{\frac{q}{p}}.
\end{aligned}$$

It is easy to see that

$$\int_s^{2^{-2j\beta}} 2^{2mj'\beta} t^{m'+p-\mu p-2} dt \lesssim 2^{2j\beta p\mu} 2^{2j\beta} 2^{-2j\beta p}.$$

Because  $s2^{2j'\beta} \leq 1$  and  $m' + p - 1 - \frac{p}{2\beta} \leq p\mu$ , one has

$$(s2^{2j'\beta})^{\frac{p}{2\beta}+1-p+p\mu} \leq (s2^{2j'\beta})^{m'}.$$

Finally, we obtain

$$\begin{aligned}
I_{Q_r}^4 &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j' \geq -\log_2 r_w} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2(j'-2)\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{p}{2\beta}+1-p+p\mu} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Subcase 6.8.2:  $q > p$ . By Hölder's inequality, we have

$$\begin{aligned}
I_{Q_r}^4 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-N} \sum_{j \geq -\log_2 r_w} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-2(j'-2)\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{p}{2\beta}+1-p+p\mu} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

## 7. PROOF OF LEMMA 5.2

7.1. **The setting**  $1 < p \leq 2$ . We divide the proof into three cases.

Case 7.1: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 1}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, I}.$$

For this case, if  $s2^{2j'\beta} \leq 1$ , then by  $v \in \mathbb{B}_{p,q,m,m'}^{\gamma_1,\gamma_2} \subset \mathbb{B}_{\frac{m}{p},\infty}^{\gamma_1-\gamma_2}$  we have  $|v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2}+\gamma_1-\gamma_2)j'}$ . By (i) of Lemma 4.2, we get

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{n}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', \epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \right. \\ &\quad \left. e^{-ct2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j''}k'' - k|)^{-N} \right\} ds \\ &\lesssim 2^{\frac{n}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{\epsilon', \epsilon'', k''} \int_0^{2^{-1-22j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)|^{p-1} \right. \\ &\quad \left. |v_{j',k''}^{\epsilon''}(s)|^{2-p} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |2^{j-j''}k'' - k|)^{-N} \right\} ds. \end{aligned}$$

Applying (4.4) and Hölder's inequality for  $s$ , we get

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{n}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w'|)^{-N} (1 + |w|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \int_0^{2^{-1-2j'\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} ds \\ &\lesssim 2^{\frac{n}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w'|)^{-N} (1 + |w|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}}. \end{aligned}$$

Because  $v \in \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ , we have

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{-nj+p\gamma_2} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})}$$

and

$$\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-1-2j'\beta}.$$

Consequently,

$$\begin{aligned} |b_{j,k}^{\epsilon,1}(t)| &\lesssim 2^{\frac{n}{2}+j} e^{-ct2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \left[ 2^{-nj+p\gamma_2} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-2j'\beta} \right]^{(1-\frac{1}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ &\lesssim 2^{\frac{n}{2}+j} e^{-ct2^{2j\beta}} 2^{j(-n+p\gamma_2)(1-\frac{1}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} 2^{-2j'\beta(1-\frac{1}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

This in turn yields

$$\begin{aligned}
I_{Q_r}^5(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{q}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \right. \\
&\quad 2^{\frac{n}{2}+j} 2^{j(-n+p\gamma_2)(1-\frac{1}{p})} e^{-ct2^{2j\beta}} \sum_{j < j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} \\
&\quad \left. \left. 2^{-2j'\beta(1-\frac{1}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{q}{p})} 2^{\frac{qm}{2}+qj} 2^{qj(-n+p\gamma_2)(1-\frac{1}{p})} \\
&\quad \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} \right. \right. \\
&\quad \left. \left. 2^{-2j'\beta(1-\frac{1}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}}.
\end{aligned}$$

Let

$$A_{j'} = \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.$$

By Hölder's inequality, we have, for any  $\delta > 0$ ,

$$\begin{aligned}
&\left[ \sum_{j < j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} 2^{-2j'\beta(1-\frac{1}{p})} A_{j'} \right]^p \\
&\lesssim \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} 2^{-2j'\beta(p-1)} A_{j'}^p.
\end{aligned}$$

On the other hand, since  $0 < s < 2^{-1-2j'\beta}$  and  $m' < 1$ , one has  $(s2^{2j'\beta})^{1-m'} \lesssim 1$ . This implies

$$\begin{aligned}
A_{j'}^p &= \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \\
&\lesssim 2^{-2j'\beta} \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s},
\end{aligned}$$

and then

$$\begin{aligned}
I_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{q}{p})} 2^{\frac{qm}{2}+qj} 2^{qj(-n+p\gamma_2)(1-\frac{1}{p})} \\
&\quad \left\{ \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{(\epsilon, k) \in S_r^j} \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} \right. \\
&\quad \left. 2^{-2pj'\beta} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right) \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 7.1.1:  $q \leq p$ . For  $p\gamma_2 + 2 - 2\beta > 0$ , take  $0 < \delta < p(p\gamma_2 + 2 - 2\beta)$ . By the  $\alpha$ -triangle inequality, we get

$$\begin{aligned}
I_{Q_r}^5(t) &\lesssim |Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj[p\gamma_2 + 2 - 2\beta]} \\
&\quad \sum_{j \leq j' + 2} 2^{\frac{q\delta(j' - j)}{p}} 2^{-q(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} 2^{-2qj'\beta} \\
&\quad \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim |Q|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \sum_{j \leq j' + 2} 2^{q(j' - j)[\frac{\delta}{p} - (p\gamma_2 + 2 - 2\beta)]} \\
&\quad 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Subcase 7.1.2:  $q > p$ . By Hölder's inequality, we get

$$\begin{aligned}
I_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ \sum_{j \leq j' + 2} 2^{(j' - j)[\delta - (p\gamma_2 + 2 - 2\beta)]} \right. \\
&\quad \left. 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right] \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left[ \sum_{j \leq j' + 2} 2^{(j' - j)[\delta - (p\gamma_2 + 2 - 2\beta)]} \right]^{\frac{q-p}{p}} \\
&\quad \times \left\{ \sum_{j \leq j' + 2} 2^{(j' - j)[\delta - (p\gamma_2 + 2 - 2\beta)]} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\
&\quad \left. \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 7.2: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, I}.$$

At first, we assume  $t > r^{2\beta}$ . For simplicity, suppose

$$\|u\|_{\mathbf{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}} = \|v\|_{\mathbf{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}} = 1.$$

If  $2^{-1-2(j'+2)\beta} < s < \frac{t}{2}$ , then by  $v \in \mathbf{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2}$  we get

$$|v_{j', k'}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2 + \frac{2m\beta}{p})j'} s^{-\frac{m}{p}}.$$

Using (4.4), Hölder's inequality and (ii) of Lemma 4.2, we can estimate  $|b_{j,k}^{\epsilon,2}(t)|$  as follows.

$$\begin{aligned}
& |b_{j,k}^{\epsilon,2}(t)| \\
& \lesssim 2^{\frac{n}{2}j+j} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2+2m\beta)j'} \sum_{\epsilon',k',\epsilon'',k''} \int_{2^{-1-2(j'+2)\beta}}^{\frac{t}{2}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)|^{p-1} \right. \\
& \quad \times s^{-m(2-p)} e^{-ct2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\
& \lesssim 2^{\frac{n}{2}j+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w,w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2+\frac{2m\beta}{p})j'} \\
& \quad \times \int_{2^{-1-2(j'+2)\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} s^{-\frac{(2-p)m}{p}} ds.
\end{aligned}$$

Now we estimate the term  $\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p$ . Notice that  $j \leq j' + 2$  implies  $s \geq 2^{-1-2j'\beta}$ . If  $s \geq 2^{-2j'\beta}$ , then by  $u \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I}$  we have

$$\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \lesssim 2^{p\gamma_2j-nj} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} (s2^{2j'\beta})^{-m}.$$

On the other hand, if  $2^{-1-2j'\beta} \leq s \leq 2^{-2j'\beta}$ , by the definition of  $\mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ , we still have

$$\begin{aligned}
\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p & \lesssim 2^{p\gamma_2j-nj} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
& \lesssim 2^{p\gamma_2j-nj} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} (s2^{2j'\beta})^{-m}.
\end{aligned}$$

Similarly, for  $v$ , we can also get

$$\sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2j-nj} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} (s2^{2j'\beta})^{-m}.$$

The above estimates yields

$$\begin{aligned}
|b_{j,k}^{\epsilon,2}(t)| & \lesssim 2^{\frac{n}{2}j+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w,w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2+\frac{2m\beta}{p})j'} \\
& \quad \times \left[ \int_{2^{-1-2(j'+2)\beta}}^{\frac{t}{2}} 2^{p\gamma_2j-nj} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} s^{-\frac{(2-p)m}{p}} (s2^{2j'\beta})^{-m} ds \right] \\
& \lesssim 2^{-\frac{n}{2}j+j+p\gamma_2j} e^{-ct2^{2j\beta}} \\
& \quad \times \left( \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2+\frac{2m\beta}{p})j'} 2^{-2j'\beta m} 2^{-j'p(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-2j'\beta(1-\frac{2m}{p})} \right) \\
& \lesssim 2^{-\frac{n}{2}j-j+2\beta j} e^{-ct2^{2j\beta}},
\end{aligned}$$

where we have used  $p\gamma_2 + 2 - 2\beta > 0$ .



Finally, we can get, by  $p\gamma_2 \leq n$ ,

$$\begin{aligned}
& I_{Q_r}^6(t) \\
&= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,2}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon, k) \in S_r^j} 2^{p(2\beta-1-\frac{n}{2})} e^{-cpt2^{2j\beta}} (t2^{2j\beta})^m \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-\frac{qnj}{2}} 2^{q(2\beta-1)j} \left[ e^{-ct2^{2j\beta}} (t2^{2j\beta})^{\frac{qm}{p}} (2^j r)^{\frac{qm}{p}} \right] \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj\gamma_2} e^{-ct2^{2j\beta}} (t2^{2j\beta})^{\frac{qm}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} t^{-\frac{q\gamma_2}{2\beta}} (t2^{2j\beta})^{\frac{qm}{p}+\frac{q\gamma_2}{2\beta}} e^{-ct2^{2j\beta}} \lesssim 1,
\end{aligned}$$

where we have used  $t > r^{2\beta}$ .

Next, we consider the case  $t < r^{2\beta}$ . Similarly, because  $v \in \mathbb{B}_{\frac{m}{p}, \infty}^{\gamma_1-\gamma_2}$ , one gets

$$|v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2}+\gamma_1-\gamma_2+\frac{2m\beta}{p})j'} s^{-\frac{m}{p}} \lesssim 2^{-j'(\frac{n}{2}+\gamma_1-\gamma_2)}.$$

An application of Hölder's inequality and (ii) of Lemma 4.2 gives

$$\begin{aligned}
& |b_{j,k}^{\epsilon,2}(t)| \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon, k'} \sum_{\epsilon'', k''} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \{ |u_{j',k'}^{\epsilon'}(t)| |v_{j',k''}^{\epsilon''}(s)| \\
&\quad \times e^{-ct2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |k' - k''|)^{-N} \} ds \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\
&\quad \times e^{-ct2^{2j\beta}} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} ds \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\
&\quad \times e^{-ct2^{2j\beta}} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}}.
\end{aligned}$$

For  $s2^{2j'\beta} \geq \frac{1}{2}$ , because  $v \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I} \cap \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II}$ , one has

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - n j} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} (s2^{2j'\beta})^{-m}$$

and

$$\int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-2j'\beta}.$$

The above estimates imply

$$\begin{aligned} |b_{j,k}^{\epsilon, 2}(t)| &\lesssim 2^{\frac{nj}{2} + j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} e^{-ct2^{2j\beta}} \\ &\quad [2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-2j'\beta}]^{1 - \frac{1}{p}} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

whence giving

$$\begin{aligned} I_{Q_r}^6(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon, 2}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} \left[ 2^{\frac{nj}{2} + j} e^{-ct2^{2j\beta}} \right. \right. \\ &\quad \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{j(p\gamma_2 - n)(1 - \frac{1}{p})} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} \\ &\quad \left. \left. 2^{-2j'\beta(1 - \frac{1}{p})} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(p\gamma_2 + 2 - 2\beta)} \left\{ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\ &\quad \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-2j'\beta(1 - \frac{1}{p})} \right. \\ &\quad \left. \left. \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}}. \end{aligned}$$

Let

$$A_{j'} = \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.$$

By Hölder's inequality, we obtain that for any  $\delta > 0$ ,

$$\begin{aligned} &\left[ \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-2j'\beta(1 - \frac{1}{p})} A_{j'} \right]^p \\ &\lesssim \sum_{j < j' + 2} 2^{\delta(j' - j)} 2^{-p(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-2j'\beta(p-1)} A_{j'}^p. \end{aligned}$$

It is easy to see that

$$\begin{aligned}
& \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{Q_{j,k}}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \\
&= 2^{-2j'\beta} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m (s2^{2j'\beta})^{1-m} \frac{ds}{s} \\
&\lesssim 2^{-2j'\beta} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s}.
\end{aligned}$$

The above estimates derive

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(p\gamma_2 + 2 - 2\beta)} \left\{ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\
&\quad \sum_{j \leq j' + 2} 2^{\delta(j' - j)} 2^{-p(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} 2^{-2pj'\beta} \\
&\quad \left. \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \right\}.
\end{aligned}$$

Subcase 7.2.1:  $q \leq p$ . Take  $0 < \delta < p(p\gamma_2 + 2 - 2\beta)$ . By the  $\alpha$ -triangle inequality, we obtain

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(p\gamma_2 + 2 - 2\beta)} \\
&\quad \sum_{j \leq j' + 2} 2^{\frac{q\delta(j' - j)}{p}} 2^{-qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})(p-1)} 2^{-2qj'\beta} 2^{-q(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \\
&\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \leq j' + 2} 2^{q(j' - j)[\frac{\delta}{p} - (p\gamma_2 + 2 - 2\beta)]} \\
&\quad 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \left( \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}} \right).
\end{aligned}$$

Subcase 7.2.2:  $q > p$ . Under this, we have

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj(p\gamma_2+2-2\beta)} \right. \\
&\quad \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{-p(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})(p-1)} 2^{-2pj'\beta} \\
&\quad \left. \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right] \right\}^{\frac{q}{p}} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ \sum_{j \leq j'+2} 2^{(j'-j)[\delta-p(p\gamma_2+2-2\beta)]} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right] \right\}^{\frac{q}{p}}.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left( \sum_{j \leq j'+2} 2^{(j'-j)[\delta-p(p\gamma_2+2-2\beta)]} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{(j'-j)[\delta-p(p\gamma_2+2-2\beta)]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \times \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim (\|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}).
\end{aligned}$$

Case 7.3: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 3}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, I}.$$

For this case, if  $\frac{t}{2} \leq s \leq t$ , then  $v \in \mathbf{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2}$  implies

$$|v_{j', k'}^{\epsilon'}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (s 2^{2j'\beta})^{-\frac{m}{p}},$$

where  $m > p$ . Then for  $|b_{j,k}^{\epsilon, 3}(t)|$ , we have, by (iii) of Lemma 4.2,

$$\begin{aligned}
|b_{j,k}^{\epsilon, 3}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| \right. \\
&\quad \left. \times e^{-(t-s)2^{2j\beta}} (1 + |k - 2^{j-j'} k'|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds \\
&\lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \sum_{\epsilon', k', \epsilon'', k''} \\
&\quad \int_{\frac{t}{2}}^t \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} (s 2^{2j'\beta})^{-\frac{m(2-p)}{p}} e^{-c(t-s)2^{2j\beta}} \right. \\
&\quad \left. \times (1 + |k - 2^{j-j'} k'|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds.
\end{aligned}$$

Applying (4.4), we obtain

$$\begin{aligned}
& |b_{j,k}^{\epsilon,3}(t)| \\
& \lesssim 2^{\frac{n_j}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} (t2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\
& \int_{\frac{t}{2}}^t \left\{ e^{-c(t-s)2^{2j\beta}} |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)|^{p-1} \right. \\
& \quad \left. \times (1 + |k - 2^{j-j'}k'|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds \\
& \lesssim 2^{\frac{n_j}{2}+j} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w-w'|)^N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} (t2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\
& \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} ds \\
& \lesssim 2^{\frac{n_j}{2}+j} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w-w'|)^N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} (t2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\
& \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}},
\end{aligned}$$

where we have used Hölder's inequality again.

Because  $v \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ , if  $s \geq 2^{-2j\beta}$ , we have

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - n j} 2^{-p j' (\frac{n}{2} + \gamma_1 - \frac{n}{p})} (s 2^{2j'\beta})^{-m}.$$

From  $\frac{t}{2} \leq s \leq t$  it follows that

$$\int_{\frac{t}{2}}^t \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{p\gamma_2 j - n j} 2^{-p j' (\frac{n}{2} + \gamma_1 - \frac{n}{p})} t (t 2^{2j'\beta})^{-m}.$$

This in turn implies

$$\begin{aligned}
|b_{j,k}^{\epsilon,3}(t)| & \lesssim 2^{\frac{n_j}{2}+j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} (t2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\
& 2^{(p\gamma_2 j - n j)(p-1)/p} 2^{-(p-1)(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} t^{\frac{p-1}{p}} (t2^{2j'\beta})^{-\frac{m(p-1)}{p}} \\
& \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\
& \lesssim 2^{\frac{n_j}{2}+j} 2^{(p\gamma_2 j - n j)(1-\frac{1}{p})} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} t^{\frac{p-1}{p}} \\
& (t2^{2j'\beta})^{-\frac{m}{p}} 2^{-(p-1)(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_{Q_r}^7(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,3}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ \sum_{(\epsilon, k) \in S_r^j} 2^{-\frac{pmj}{2} + pj + p(p-1)\gamma_2 j + nj} \right. \\
&\quad \left[ \sum_{j \leq j' + 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-[\frac{n}{2} + \gamma_1 + \gamma_2(p-2)]j'} 2^{\frac{(p-1)nj'}{p}} \right. \\
&\quad \left. \left( \int_{\frac{t}{2}}^t e^{-(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{r,j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} (t2^{2j'\beta})^{-\frac{m}{p}} \right]^p t^{p-1} (t2^{2j\beta})^m \Big\}^{\frac{q}{p}}.
\end{aligned}$$

For any  $\delta > 0$ , we use (4.5) to get

$$\begin{aligned}
I_{Q_r}^7(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ 2^{-\frac{pmj}{2} + pj + p(p-1)\gamma_2 j + nj} \right. \\
&\quad \sum_{j \leq j' + 2} 2^{\delta(j'-j)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{(\epsilon', k') \in S_{r,j,k}^{w,j'}} 2^{-p[\frac{n}{2} + \gamma_1 + \gamma_2(p-2)]j'} 2^{(p-1)nj'} \\
&\quad \left. \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} |u_{j',k'}^{\epsilon'}(s)|^p ds (t2^{2j'\beta})^{-m} t^{p-1} (t2^{2j\beta})^m \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 7.3.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned}
I_{Q_r}^7(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad 2^{-\frac{qnj}{2} + qj + q(p-1)\gamma_2 j + \frac{qmj}{p}} \sum_{j \geq j' + 2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-q[\frac{n}{2} + \gamma_1 + (p-2)\gamma_2]j'} 2^{\frac{q(p-1)nj'}{p}} \\
&\quad \left[ \int_{\frac{t}{2}}^t e^{-(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{r,j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right]^{\frac{q}{p}} (t2^{2j'\beta})^{-\frac{qm}{p}} t^{\frac{q(p-1)}{p}} (t2^{2j\beta})^{\frac{qm}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj[p\gamma_2 + 2(1-\beta)]} \\
&\quad \sum_{j \geq j' + 2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-qj'[2\gamma_1 + (p-2)\gamma_2]} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left[ \int_{\frac{t}{2}}^t e^{-(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{r,j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right]^{\frac{q}{p}} (t2^{2j'\beta})^{-\frac{qm}{p}} t^{\frac{q(p-1)}{p}} (t2^{2j\beta})^{\frac{qm}{p}}.
\end{aligned}$$

Because  $u \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ , for a fixed  $j'$  we have

$$|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon', k') \in S_{r,j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \right]^{\frac{q}{p}} \lesssim 1.$$

This in turn implies

$$\begin{aligned}
I_{Q_r}^7(t) &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj[p\gamma_2 + 2(1-\beta)]} \\
&\quad \sum_{j \leq j' + 2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-qj'[2\gamma_1 + (p-2)\gamma_2]} t^{\frac{q}{p}} (t2^{2j'\beta})^{-\frac{2qm}{p}} t^{\frac{q(p-1)}{p}} (t2^{2j\beta})^{\frac{qm}{p}} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj[p\gamma_2 + 2(1-\beta)]} \\
&\quad \left\{ \sum_{j \leq j' + 2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-qj'[p\gamma_2 + 2(1-\beta)]} 2^{2qj'\beta} 2^{-\frac{4j'\beta qm}{p}} \right\} t^{q - \frac{qm}{p}} (2^{2j\beta})^{\frac{qm}{p}} \\
&\lesssim \sum_{j \geq \max\{-\log_2 t, -\frac{\log_2 t}{2\beta}\}} 2^{qj[p\gamma_2 + 2(1-\beta)]} \\
&\quad \times \left\{ \sum_{j \leq j' + 2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-qj'[p\gamma_2 + 2(1-\beta)]} 2^{2q\beta(j'-j)} 2^{\frac{4\beta m q(j-j')}{p}} \right\} (t2^{2j\beta})^{\frac{q(p-m)}{p}} \\
&= \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} (t2^{2j\beta})^{\frac{q(p-m)}{p}} \sum_{j \leq j' + 2} 2^{q(j-j')[p\gamma_2 + 2(1-\beta) - 2\beta + \frac{4\beta m}{p} - \delta]}.
\end{aligned}$$

Because  $m > p$  and  $t2^{2j\beta} \geq 1$ , we have

$$\sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} (t2^{2j\beta})^{\frac{q(p-m)}{p}} \lesssim 1.$$

Notice that  $p\gamma_2 + 2 - 2\beta > 0$ . If

$$0 < \delta < p\gamma_2 + 2(1-\beta) - 2\beta + \frac{4\beta m}{p},$$

we can obtain

$$I_Q^7 \lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, I}} + \|u\|_{\mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}}.$$

Subcase 7.3.2:  $q > p$ . By Hölder's inequality, we get

$$\begin{aligned}
I_{Q_r}^7(t) &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj[p\gamma_2 + 2(1-\beta)]} \right. \\
&\quad \sum_{j \leq j' + 2} 2^{\delta(j'-j)} 2^{-pj'[p\gamma_2 + 2(1-2\beta)]} \left[ \int_{\frac{t}{2}}^t e^{-(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right] \\
&\quad \left. (t2^{2j'\beta})^{-m} \right\} t^{\frac{q}{p}} t^{\frac{q(p-1)}{p}} (t2^{2j\beta})^{\frac{qm}{p}}.
\end{aligned}$$

Because

$$\sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \lesssim |Q_r|^{-p(\frac{\gamma_2}{n} - \frac{q}{p})} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})},$$

we have

$$\begin{aligned}
& I_{Q_r}^7(t) \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj[p\gamma_2+2(1-\beta)]} \sum_{j \leq j'+2} 2^{\delta(j'-j)} \right. \\
& \quad \left. 2^{-pj'[p\gamma_2+2(1-2\beta)]} \left[ \int_{\frac{t}{2}}^t ds \right] (t2^{2j'\beta})^{-m} \right\}^{\frac{q}{p}} t^{\frac{q(p-1)}{p}} (t2^{2j\beta})^{\frac{qm}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj[p\gamma_2+2(1-\beta)]} \sum_{j \leq j'+2} 2^{\delta(j'-j)} \right. \\
& \quad \left. 2^{-pj'[p\gamma_2+2(1-2\beta)]} t(t2^{2j'\beta})^{-m} \right\}^{\frac{q}{p}} t^{\frac{q(p-1)}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ \sum_{j \leq j'+2} 2^{q(j-j')[p\gamma_2+2-4\beta+\frac{4\beta m}{p}-\delta]} (t2^{2j\beta})^{p-m} \right\}^{\frac{q}{p}} \\
& \lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,I}} + \|u\|_{\mathbf{B}_{p,q}^{\gamma_1,\gamma_2,II}}.
\end{aligned}$$

**7.2. The setting**  $2 < p < \infty$ . We also divide the proof into three parts.

Case 7.4: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon,1}(t) \Phi_{j,k}^\epsilon(x) \text{ is in } \mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,I}.$$

By (i) of Lemma 4.2, we apply Hölder's inequality and (4.6) to see

$$\begin{aligned}
|b_{j,k}^{\epsilon,1}(t)| & \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \right. \\
& \quad \left. (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds \\
& \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
& \quad \int_0^{2^{-1-2j'\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
& \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
& \quad \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-1-2j'\beta}} ds \right)^{1-\frac{2}{p}} \\
& \quad \times \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Because  $0 < s < 2^{-1-2j'\beta}$ ,  $v \in \mathbf{B}_{p,q}^{\gamma_1,\gamma_2,II}$  implies

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{p\gamma_2 j - n j - p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}.$$

This in turn implies

$$\left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}} \lesssim 2^{\gamma_2 j - \frac{n_j}{p} - j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}}.$$



Thus

$$|b_{j,k}^{\epsilon,1}(t)| \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} 2^{-2j'\beta(1-\frac{2}{p})} \\ 2^{\gamma_2 j - \frac{n_j}{p}} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.$$

Using the above estimate, we obtain

$$I_{Q_r}^5(t) = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon,k) \in S_r^j} \left[ 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \right. \right. \\ \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} 2^{-2j'\beta(1-\frac{2}{p})} 2^{j(\gamma_2 - \frac{n}{p})} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ \left. 2^{-\frac{2j'\beta}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \Big\}^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{\frac{qn_j}{2} + qj} 2^{qj(\gamma_2 - \frac{n}{p})} \\ \left\{ \sum_{(\epsilon,k) \in S_r^j} e^{-cpt2^{2j\beta}} (t2^{2j\beta})^m \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \left[ \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} 2^{-j'\beta(2-\frac{4}{p})} \right. \right. \\ \left. \left. 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{-\frac{2j'\beta}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}}.$$

Thanks to  $0 < s < 2^{-1-2j'\beta}$  and  $m' < 1$ , one gets

$$\int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \\ \lesssim 2^{-2j'\beta} \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s}.$$

Also, for any  $0 < \delta < (\gamma_1 + \gamma_2 + 1)$ , we have

$$I_{Q_r}^5(t) \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \gamma_2 + 1 + n - \frac{2n}{p})} \\ \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{(\delta + pn - 2n)(j'-j)} 2^{-2j'\beta(p-2)} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ \left. 2^{-4j'\beta} \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right] \right\}^{\frac{q}{p}}.$$

Subcase 7.4.1:  $q \leq p$ . The  $\alpha$ -triangle inequality implies

$$\begin{aligned}
I_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \\
&\quad \sum_{j \leq j'+2} 2^{\frac{q\delta(j'-j)}{p}} 2^{qn(j'-j)(1-\frac{2}{p})} 2^{-2qj'\beta(1-\frac{2}{p})} 2^{-qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad 2^{-2qj'\beta/p} 2^{-2qj'\beta/p} \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \sum_{j < j'+2} 2^{q(\frac{\delta}{p}-\gamma_1-\gamma_2-1)(j'-j)} \\
&\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Subcase 7.4.2:  $q > p$ . By Hölder's inequality, we obtain

$$\begin{aligned}
I_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \right. \\
&\quad \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-pj'(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left. \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left( \sum_{j < j'+2} 2^{p(j'-j)(\frac{\delta}{p}-\gamma_1-\gamma_2-1)} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j < j'+2} 2^{(j'-j)[\delta-p(\gamma_1+\gamma_2+1)]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 7.5: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, I}.$$

Without loss of generality, we may assume

$$\|u\|_{\mathbf{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}} = \|v\|_{\mathbf{B}_{p, q, m, m'}^{\gamma_1, \gamma_2}} = 1.$$

Suppose firstly  $t > r^{2\beta}$ . We have, using (4.6) and (ii) of Lemma 4.2,

$$|b_{j,k}^{\epsilon,2}(t)| \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds.$$

Due to

$$u, v \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,I} \cap \mathbb{B}_{p,q}^{\gamma_1,\gamma_2,II},$$

for  $2^{-1-2j'\beta} < s < \frac{t}{2}$  we have

$$\sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (s2^{2j'\beta})^{-m}$$

and

$$\sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (s2^{2j'\beta})^{-m}.$$

Notice that  $\gamma_1 + \gamma_2 + 1 > 0$  is equivalent to  $\gamma_1 > -\beta$ . So, we have

$$|b_{j,k}^{\epsilon,2}(t)| \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \left[ 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right]^{\frac{2}{p}} \\ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} (s2^{2j'\beta})^{-\frac{2m}{p}} ds \\ \lesssim 2^{\frac{n_j}{2}+j} e^{-ct2^{2j\beta}} 2^{2\gamma_2 j - \frac{2nj}{p}} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} 2^{-2j'(\gamma_1 + \frac{n}{2} - \frac{n}{p} + \frac{2m\beta}{p})} 2^{-2j'\beta(1-\frac{2m}{p})} \\ \lesssim 2^{j(2\gamma_1 - \frac{n}{2} + 1)} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} 2^{-2j'(\gamma_1 + \beta)} \\ \lesssim 2^{-\frac{n_j}{2}} 2^{j(2\gamma_2 - 2\gamma_1 - 2\beta + 1)} e^{-ct2^{2j\beta}}.$$

The above estimate implies

$$I_{Q_r}^6(t) \\ = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,2}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ (2^j r)^n 2^{pj(2\beta - 1 - \frac{n}{2})} e^{-ct2^{2j\beta}} (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} t^{-\frac{q\gamma_2}{2\beta}} (t2^{2j\beta})^{\frac{q\gamma_2 j}{2\beta}} e^{-ct2^{2j\beta}} (t2^{2j\beta})^{\frac{qm}{p}} \\ \lesssim 1,$$

where we have used  $t > r^{2\beta}$ .

Suppose secondly  $t \leq r^{2\beta}$ . Similarly, we can obtain, by (ii) of Lemma 4.2,

$$\begin{aligned}
|b_{j,k}^{\epsilon,2}(t)| &\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{1}{p})} \\
&\quad \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
&\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \\
&\quad \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{1}{p})} (2^{-2j'\beta})^{\frac{p-2m}{p}} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p s^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \\
&\quad \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{1}{p})} 2^{-2j'\beta} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

Since  $v \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III} \cap \mathbb{B}_{p,q,IV}^{\gamma_1,\gamma_2,IV}$ , we have

$$\begin{aligned}
&\int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \\
&\leq \int_{2^{-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \\
&\quad + \int_{2^{-1-2j'\beta}}^{2^{-2j'\beta}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \\
&\lesssim 2^{p\gamma_2 j - n j - p j' (\gamma_1 + \frac{n}{2} - \frac{n}{p})},
\end{aligned}$$

whence getting

$$\begin{aligned}
|b_{j,k}^{\epsilon,2}(t)| &\lesssim 2^{\frac{nj}{2}+j} 2^{\gamma_2 j - \frac{nj}{p}} e^{-ct2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad 2^{-2j'\beta} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

By Hölder's inequality, we have for any  $\delta > 0$ ,

$$\begin{aligned}
I_{Q_r}^6(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,2}(t)|^p (t2^{2j\beta})^m \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} (t2^{2j\beta})^m \right. \\
&\quad \left[ 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{n(j'-j)(1-\frac{2}{p})} 2^{-2j'\beta} 2^{\gamma_2 j - \frac{nj}{p}} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \left. \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \\
&\quad \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-2pj'\beta} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right] \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 7.5.1:  $q \leq p$ . Take  $0 < \delta < \gamma_1 + \gamma_2 + 1$ . By the  $\alpha$ -triangle inequality, we get

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \\
&\quad \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \leq j'+2} 2^{\frac{q\delta(j'-j)}{p}} 2^{\frac{qn(j'-j)(p-2)}{p}} 2^{-2qj'\beta} 2^{-qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \\
&\quad \sum_{j \leq j'+2} 2^{q(j'-j)[\frac{\delta}{p}-(\gamma_1+\gamma_2+1)]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

Subcase 7.5.2:  $q > p$ . Take  $0 < \delta < \gamma_1 + \gamma_2 + 1$ . The Hölder inequality implies

$$\begin{aligned}
I_{Q_r}^6(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{pj(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \right. \\
&\quad \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-2pj'\beta} 2^{-pj'(\gamma_1+\gamma_2+1+n-\frac{2n}{p})} \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left( \sum_{j \leq j'+2} 2^{(j'-j)[\delta-p(\gamma_1+\gamma_2+1)]} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{(j'-j)[\delta-p(\gamma_1+\gamma_2+1)]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \times \left[ \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 7.6: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 3}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, I}.$$

For simplicity, we may once again assume

$$\|u\|_{\mathbf{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}} = \|v\|_{\mathbf{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}} = 1.$$

By (4.6) and (iii) of Lemma 4.2, we get

$$\begin{aligned}
|b_{j,k}^{\epsilon, 3}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k'} \sum_{\epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| e^{-c(t-s)2^{2j\beta}} \right. \\
&\quad \left. \times (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds \\
&\lesssim 2^{\frac{nj}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds.
\end{aligned}$$

Because  $v \in \mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, I} \cap \mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}$ , we have, for  $\frac{t}{2} \leq s \leq t$ ,

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (t 2^{2j'\beta})^{-m}$$

and

$$\left( \int_{\frac{t}{2}}^t \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}} \lesssim t^{\frac{1}{p}} 2^{\gamma_2 j - \frac{nj}{p}} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (t 2^{2j'\beta})^{-\frac{m}{p}}.$$

Consequently,

$$|b_{j,k}^{\epsilon,3}(t)| \lesssim 2^{\frac{n_j}{2}+j} 2^{\gamma_2 j - \frac{n_j}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ 2^{n(j'-j)(1-\frac{2}{p})} t^{\frac{1}{p}} (t 2^{2j'\beta})^{-\frac{m}{p}} t^{\frac{p-2}{p}} \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.$$

The above estimate in turn implies

$$I_{Q_r}^7(t) = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,3}(t)|^p (t 2^{2j\beta})^m \right]^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon,k) \in S_r^j} 2^{\frac{pn_j}{2} + pj + p\gamma_2 j} 2^{-nj} \right. \\ \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (t 2^{2j'\beta})^{-\frac{m}{p}} \right. \\ \left. \left. 2^{n(j'-j)(1-\frac{2}{p})} \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p t^{p-1} (t 2^{2j\beta})^m \right\}^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ 2^{\frac{pn_j}{2} + pj + p\gamma_2 j} 2^{-nj} \right. \\ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (t 2^{2j'\beta})^{-m} 2^{n(j'-j)(p-2)} \\ \left. \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) t^{p-1} (t 2^{2j\beta})^m \right\}^{\frac{q}{p}}.$$

In a similar manner,  $u \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I} \cap \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$  yields

$$\sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \lesssim |Q_r|^{1-\frac{p\gamma_2}{n}} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (s 2^{2j'\beta})^{-m}.$$

Because  $\gamma_1 + \gamma_2 + 1 > 0$ , we have  $2p\gamma_1 + 4m\beta > 0$ . Upon taking  $0 < \delta < 2p\gamma_1 + 4m\beta$ , we achieve

$$I_{Q_r}^7(t) \lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq \max\{-\log_2 r, -\frac{\log_2 t}{2\beta}\}} \left\{ 2^{j[2p\gamma_2 + 2p(1-\beta) + (p-2)n]} \right. \\ \left. \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-2pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (t 2^{2j'\beta})^{-2m} (t 2^{2j\beta})^m t^p \right\}^{\frac{q}{p}} \\ \lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\frac{\log_2 t}{2\beta}} \left\{ \sum_{j \leq j'+2} 2^{pj'[\gamma_1 + \gamma_2 + 1 + n - \frac{2n}{p} + \frac{2m\beta}{p}]} 2^{-2j\beta(p-m)} \right. \\ \left. 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-pj'(2\gamma_1 + n - \frac{2n}{p} + \frac{4m\beta}{p})} \right\}^{\frac{q}{p}} (t 2^{2j\beta})^{q(1-\frac{m}{p})} \\ \lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\frac{\log_2 t}{2\beta}} \left\{ \sum_{j \leq j'+2} 2^{(j'-j)[\delta - 2p\gamma_1 - \frac{4m\beta}{p}]} \right\}^{\frac{q}{p}} (t 2^{2j\beta})^{q(1-\frac{m}{p})} \\ \lesssim 1,$$

where we have used  $p < m$ .

## 8. PROOF OF LEMMA 5.3

8.1. **The setting**  $1 < p \leq 2$ . The argument is split into three cases.

Case 8.1: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon,1}(t) \Phi_{j,k}^\epsilon(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}.$$

Because  $u$  and  $v$  both belong to  $\mathbb{B}_{\frac{m}{p}, \infty}^{\gamma_1 - \gamma_2}$ , for  $s2^{2j'\beta} \leq 1$  we have

$$|u_{j',k'}^{\epsilon'}(s)| + |v_{j',k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'}.$$

By Hölder's inequality, (4.4) and (i) of Lemma 4.2, we get

$$\begin{aligned} & |b_{j,k}^{\epsilon,1}(t)| \\ & \lesssim 2^{\frac{nj}{2} + j} e^{-ct2^{2j\beta}} \sum_{j \leq j' + 2} \sum_{\epsilon', k', \epsilon'', k''} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j',k'}^{\epsilon'}(s)| \right. \\ & \quad \times |v_{j',k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\ & \lesssim 2^{\frac{nj}{2} + j} e^{-ct2^{2j\beta}} \sum_{j \leq j' + 2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \\ & \quad \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}. \end{aligned}$$

Owing to  $v \in \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}$ , one has

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - n j} 2^{-p j' (\gamma_1 + \frac{n}{2} - \frac{n}{p})} \text{ a fixed } j'.$$

Applying the above estimate, we have

$$\begin{aligned} & |b_{j,k}^{\epsilon,1}(t)| \\ & \lesssim 2^{\frac{nj}{2} + j} e^{-ct2^{2j\beta}} \sum_{j \leq j' + 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \\ & \quad 2^{\frac{(p-1)(p\gamma_2 j - n j)}{p}} 2^{-(p-1)j' (\gamma_1 + \frac{n}{2} - \frac{n}{p} + \frac{2\beta}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ & \lesssim 2^{[\frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2 + 1]j} e^{-ct2^{2j\beta}} \sum_{j \leq j' + 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \\ & \quad 2^{[\frac{n}{2} - \frac{n}{p} + \frac{2\beta}{p} - (p-1)\gamma_2 - 1]j'} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$



This last estimate for  $|b_{j,k}^{\epsilon,1}(t)|$  ensures

$$\begin{aligned}
I_{Q_r}^5 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} \left[ 2^{\lfloor \frac{n}{p}-\frac{n}{2}+(p-1)\gamma_2+1 \rfloor j} \right. \right. \\
&\quad \left. \left. e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{\lfloor \frac{n}{2}-\frac{n}{p}+\frac{2\beta}{p}-(p-1)\gamma_2-1 \rfloor j'} \right. \right. \\
&\quad \left. \left. \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{qj \lfloor \frac{n}{p}-\frac{n}{2}+(p-1)\gamma_2+1 \rfloor} \\
&\quad \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} e^{-ct2^{2j\beta}} \left[ \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{\lfloor \frac{n}{2}-\frac{n}{p}+\frac{2\beta}{p}-(p-1)\gamma_2-1 \rfloor j'} \right. \right. \\
&\quad \left. \left. \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Also, for any  $\delta > 0$ , we use (4.5) to derive

$$\begin{aligned}
I_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} 2^{\delta(j'-j)} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \left. \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} 2^{pj' \lfloor \frac{n}{2}-\frac{n}{p}+\frac{2\beta}{p}-(p-1)\gamma_2-1 \rfloor} |u_{j',k'}^{\epsilon'}(s)|^p ds (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \\
&\quad \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-ct2^{2j\beta}} (t2^{2j\beta})^m \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{j' \lfloor 2\beta-p(p-1)\gamma_2-p-p\gamma_1 \rfloor} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{ds}{s}} 2^{-2j'\beta} \frac{ds}{s} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Because  $s2^{2j'\beta} \leq 1$  and  $m' < 1$ , one has  $(s2^{2j'\beta}) \leq (s2^{2j'\beta})^{m'}$ . Note also that

$$\int_{2^{-2j\beta}}^{r^{2\beta}} e^{-ct2^{2j\beta}} (t2^{2j\beta})^m \frac{dt}{t} \lesssim 1.$$

So, we arrive at

$$\begin{aligned}
I_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \left\{ \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{-pj'(p\gamma_2+2-2\beta)} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 8.1.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned}
I_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \sum_{j \leq j'+2} 2^{\frac{q\delta(j'-j)}{p}} 2^{-qj'(p\gamma_2+2-2\beta)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j'+2} 2^{q(j-j')(p\gamma_2+2-2\beta-\frac{\delta}{p})} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}}.
\end{aligned}$$

Changing the order of  $j$  and  $j'$ , we find  $I_{Q_r}^5 \lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}$ .

Subcase 8.1.2:  $q > p$ . By Hölder's inequality, we get

$$\begin{aligned}
I_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \left[ \sum_{j \leq j'+2} 2^{p(j-j')(p\gamma_2+2-2\beta-\delta)} \right. \\
&\quad \left. \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j'+2} 2^{p(j-j')(p\gamma_2+2-2\beta-\delta)} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{q(j-j')(p\gamma_2+2-2\beta-\delta)} \left[ \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \right\}.
\end{aligned}$$

Upon taking  $0 < \delta < p\gamma_2 + 2 - 2\beta$  and changing the order of  $j$  and  $j'$ , we reach  $I_{Q_r}^5 \lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}$ .

Case 8.2: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}.$$

For  $s \geq 2^{-1-2j'\beta}$ , one has

$$|v_{j', k'}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} s^{-\frac{m}{p}}.$$

This, along with (4.4) and (ii) of Lemma 4.2 yields

$$\begin{aligned}
& |b_{j,k}^{\epsilon,2}(t)| \\
& \lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \\
& \quad 2^{-(2-p)(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{-\frac{m(2-p)}{p}} |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)|^{p-1} ds \\
& \lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} \\
& \quad \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{-\frac{(2-p)m}{p}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |u_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} ds \\
& \lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} \\
& \quad \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{-m} \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p s^m \right)^{\frac{1}{p'}} ds.
\end{aligned}$$

Thanks to

$$\int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \lesssim 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})},$$

an application of Hölder's inequality gives

$$\begin{aligned}
& |b_{j,k}^{\epsilon,2}(t)| \\
& \lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \sum_{j \leq j'+2} 2^{-(2-p)(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} \\
& \quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \right)^{\frac{1}{p}} 2^{-\frac{2mj'\beta}{p'}} \\
& \quad \times \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p'}} \\
& \lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-(2-p)(\frac{n}{2} + \frac{2m\beta}{p} + \gamma_1 - \gamma_2)j'} 2^{-\frac{2mj'\beta}{p'}} \\
& \quad \times \left[ 2^{p\gamma_2 j - nj} 2^{-pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right]^{\frac{1}{p'}} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \right)^{\frac{1}{p}}.
\end{aligned}$$

By a simple calculation, we have

$$\begin{aligned}
|b_{j,k}^{\epsilon,2}(t)| &\lesssim 2^{j[\frac{n}{p}-\frac{n}{2}+1+(p-1)\gamma_2]} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \\
&\quad 2^{-(2-p)[\frac{n}{2}+\frac{2m\beta}{p}+\gamma_1-\gamma_2]j'} 2^{-j'(p-1)(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{-\frac{2m\beta j'(p-1)}{p}} \\
&\quad \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \right)^{\frac{1}{p}} \\
&\lesssim 2^{j[\frac{n}{p}-\frac{n}{2}+1+(p-1)\gamma_2]} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \\
&\quad 2^{j'[-\frac{2m\beta}{p}-\gamma_1+(2-p)\gamma_2+\frac{n}{2}-\frac{n}{p}]} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Note that  $Q_{j,k} \subset Q_r$  ensures  $j \geq -\log_2 r$ . Thus, for any  $\delta > 0$ , the last estimate for  $|b_{j,k}^{\epsilon,2}(t)|$  and (4.5) imply

$$\begin{aligned}
I_{Q_r}^6 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,2}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{pj[\frac{n}{p}-\frac{n}{2}+1+(p-1)\gamma_2]} e^{-ct2^{2j\beta}} \right. \\
&\quad \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{j'[-\frac{2m\beta}{p}-\gamma_1+(2-p)\gamma_2+\frac{n}{2}-\frac{n}{p}]} \right. \\
&\quad \left. \left. \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-ct2^{2j\beta}} (t2^{2j\beta})^m 2^{pj[\frac{n}{p}-\frac{n}{2}+1+(p-1)\gamma_2]} \right. \\
&\quad \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{\delta(j'-j)} 2^{pj'[-\frac{2m\beta}{p}-\gamma_1+(2-p)\gamma_2+\frac{n}{2}-\frac{n}{p}]} \\
&\quad \left. \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p s^{p-m-1} ds \frac{dt}{t} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Because  $s2^{2j'\beta} \geq \frac{1}{2}$  and  $m > p > 1$ , we get  $(s2^{2j'\beta})^{p-2m} \lesssim 1$  and

$$\begin{aligned}
I_{Q_r}^6 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \left\{ \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{pj'[\frac{n}{2}-\frac{n}{p}-\gamma_1+(2-p)\gamma_2-\frac{2m\beta}{p}]} \right. \\
&\quad \left. 2^{-2j'\beta(p-m)} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \left\{ \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. 2^{pj'(2\beta-2-p\gamma_2)} \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 8.2.1:  $q \leq p$ . Take  $0 < \delta < p\gamma_2 + 2 - 2\beta$ . The  $\alpha$ -triangle inequality implies

$$\begin{aligned} I_{Q_r}^6 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j' + 2} 2^{q(j-j')(p\gamma_2 + 2 - 2\beta - \frac{\delta}{p})} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\ &\quad \left[ \int_{2^{-1-2j'\beta}}^{\frac{r^{2\beta}}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\ &\lesssim \|u\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}. \end{aligned}$$

Subcase 8.2.2:  $q > p$ . Take  $0 < \delta < p\gamma_2 + 2 - 2\beta$ . By Hölder's inequality, we get

$$\begin{aligned} I_{Q_r}^6 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left\{ \sum_{j \leq j' + 2} 2^{p(j-j')(p\gamma_2 + 2 - 2\beta - \delta)} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\ &\quad \left. \int_{2^{-1-2j'\beta}}^{\frac{r^{2\beta}}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j' + 2} 2^{p(j-j')(p\gamma_2 + 2 - 2\beta - \delta)} \right)^{\frac{q-p}{p}} \\ &\quad \left\{ \sum_{j \leq j' + 2} 2^{p(j-j')(p\gamma_2 + 2 - 2\beta - \delta)} \left[ \int_{2^{-1-2j'\beta}}^{\frac{r^{2\beta}}{2}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\ &\lesssim \|u\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}. \end{aligned}$$

Case 8.3: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 3}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}.$$

Since  $s 2^{2j'\beta} \geq 1$  and  $v \in \mathbf{B}_{\frac{m}{p}, \infty}^{\gamma_1, \gamma_2}$ , one has

$$|v_{j', k''}^{\epsilon''}(s)| \lesssim 2^{-\frac{n_{j'}}{2}} 2^{-j'(\gamma_1 - \gamma_2)} (s 2^{2j'\beta})^{-\frac{m}{p}}.$$

This, via  $\frac{t}{2} \leq s \leq t$ , Hölder's inequality and (iii) of Lemma 4.2, derives

$$\begin{aligned} &|b_{j, k}^{\epsilon, 3}(t)| \\ &\lesssim 2^{\frac{n_j}{2} + j} \sum_{j \leq j' + 2} \sum_{\epsilon', k', \epsilon'', k''} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \\ &\quad 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (t 2^{2j'\beta})^{-\frac{m(2-p)}{p}} \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} ds \\ &\lesssim 2^{\frac{n_j}{2} + j} \sum_{j \leq j' + 2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (t 2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\ &\quad \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p'}} ds \\ &\lesssim 2^{\frac{n_j}{2} + j} \sum_{j \leq j' + 2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (t 2^{2j'\beta})^{-\frac{m(2-p)}{p}} \\ &\quad \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_{\frac{t}{2}}^t \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}}, \end{aligned}$$

where we have used (4.4) again.

Because of  $v \in \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}$ , one has

$$\int_{\frac{t}{2}}^t \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{p\gamma_2 j - n j} 2^{-p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} t (t 2^{2j' \beta})^{-m}.$$

This in turn implies

$$\begin{aligned} & |b_{j,k}^{\epsilon,3}(t)| \\ & \lesssim 2^{\frac{n}{2}j + j} \sum_{j \leq j' + 2} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (t 2^{2j' \beta})^{-\frac{m(2-p)}{p}} \\ & \quad \left[ 2^{pj(\gamma_2 - n)} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} t (t 2^{2j' \beta})^{-m} \right]^{\frac{p-1}{p}} \\ & \quad \times \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ & \lesssim 2^{j[-\frac{n}{2} + 1 + \frac{n}{p} + (p-1)\gamma_2]} \sum_{j \leq j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-(p-1)(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'} \\ & \quad \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} t^{\frac{p-1}{p}} (t 2^{2j' \beta})^{-\frac{m}{p}} \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

By the above estimate and (4.5), we get that if  $\delta > 0$  then

$$\begin{aligned} I_{Q_r}^7 &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,3}(t)|^p (t 2^{2j\beta})^m \frac{dt}{t} \right]^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{pj[-\frac{n}{2} + 1 + \frac{n}{p} + (p-1)\gamma_2]} \right. \\ &\quad \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-(p-1)(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} \right. \\ &\quad \left. (t 2^{2j' \beta})^{-\frac{m}{p}} \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p t^{p-1} (t 2^{2j\beta})^m \frac{dt}{t} \left. \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \right. \\ &\quad \sum_{j \leq j' + 2} 2^{-p(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} 2^{-p(p-1)(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} (t 2^{2j' \beta})^{-m} 2^{\delta(j' - j)} \\ &\quad \left. \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) t^{p-1} (t 2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}}. \end{aligned}$$

Because of  $\frac{t}{2} \leq s \leq t$ , we have

$$\begin{aligned}
I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{-pj'(2\gamma_1-2\gamma_2+p\gamma_2)} 2^{\delta(j'-j)} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \int_{2^{-2j\beta}}^{r^{2\beta}} t^{p-1} \right. \\
&\quad \times \left( \int_{\frac{t}{2}}^t e^{-cp(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j'\beta})^{-m} (t2^{2j\beta})^m \frac{dt}{t} \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{-pj'(2\gamma_1-2\gamma_2+p\gamma_2)} 2^{\delta(j'-j)} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \int_{2^{-2j\beta}}^{r^{2\beta}} t^p (t2^{2j'\beta})^{-2m} \right. \\
&\quad \times \left( \int_{\frac{t}{2}}^t e^{-cp(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) (t2^{2j\beta})^m \frac{dt}{t} \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{-pj'(2\gamma_1-2\gamma_2+p\gamma_2)} 2^{\delta(j'-j)} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \int_{2^{-1-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right. \\
&\quad \times \left[ \int_s^{2s} e^{-cp(t-s)2^{2j\beta}} t^p (t2^{2j'\beta})^{-2m} (t2^{2j\beta})^m \frac{dt}{t} \right] (s2^{2j'\beta})^m \frac{ds}{s} \Big\}^{\frac{q}{p}}.
\end{aligned}$$

Taking  $p < m$  and  $2^{-2j\beta} < s < r^{2\beta}$  into account, we can get

$$\begin{aligned}
&\int_s^{2s} e^{-cp(t-s)2^{2j\beta}} t^p (t2^{2j'\beta})^{-2m} (t2^{2j\beta})^m \frac{dt}{t} \\
&\lesssim 2^{-4j'\beta m} 2^{2j\beta m} 2^{2j\beta(m-p)} \int_s^{2s} e^{-cp(t-s)2^{2j\beta}} (t2^{2j\beta})^{p-m} \frac{dt}{t} \\
&\lesssim 2^{-4j'\beta m} 2^{2j\beta m} 2^{2j\beta(m-p)},
\end{aligned}$$

thereby finding

$$\begin{aligned}
I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(p\gamma_2+2-2\beta)} \\
&\quad \left[ \int_{2^{-2(j'+2)\beta-1}}^{r^{2\beta}} \sum_{j \leq j'+2} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{\delta(j'-j)} 2^{-pj'(p\gamma_2+2-2\beta)} \right. \\
&\quad \left. \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 8.3.1:  $q \leq p$ . Take  $0 < \frac{\delta}{p} < p\gamma_2 + 2 - 2\beta$ . By the  $\alpha$ -triangle inequality, we can get

$$\begin{aligned}
I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{q(j-j')(p\gamma_2+2-2\beta-\frac{\delta}{p})} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left[ \int_{2^{-2(j'+2)\beta-1}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta}) \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}},
\end{aligned}$$

where we have changed the order of  $j$  and  $j'$ .

Subcase 8.3.2:  $q > p$ . For this, we choose  $0 < \delta < p\gamma_2 + 2 - 2\beta$ . Then, Hölder's inequality implies

$$\begin{aligned} I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j'+2} 2^{p(j-j')(p\gamma_2+2-2\beta-\delta)} \right)^{\frac{q-p}{p}} \left\{ \sum_{j \leq j'+2} 2^{p(j-j')(p\gamma_2+2-2\beta-\delta)} \right. \\ &\quad \left. 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_{2^{-2(j'+2)\beta-1}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{\frac{q}{p}} ds \right]^{\frac{q}{p}} \right\} \\ &\lesssim \|u\|_{\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}. \end{aligned}$$

This completes the proof of Lemma 5.3 for the case  $1 < p \leq 2$ .

8.2. **The setting  $2 < p < \infty$ .** We divide the proof into three cases.

Case 8.4: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 1}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}.$$

For  $s 2^{2j'\beta} \leq 1$ , one has

$$|u_{j', k'}^{\epsilon'}(s)| + |v_{j', k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'}.$$

We can get, by (4.6) and (i) of Lemma 4.2,

$$\begin{aligned} &|b_{j, k}^{\epsilon, 1}(t)| \\ &\lesssim 2^{\frac{n}{2} + j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_0^{2^{-1-2j'\beta}} \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| \right. \\ &\quad \times e^{-ct 2^{2j\beta}} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\ &\lesssim 2^{\frac{n}{2} + j} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{n(j'-j)(1-\frac{2}{p})} \\ &\quad \int_0^{2^{-1-2j'\beta}} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\ &\lesssim 2^{\frac{n}{2} + j} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-2j'\beta(1-\frac{2}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \\ &\quad \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Since  $v \in \mathbb{B}_{p, q}^{\gamma_1, \gamma_2, II}$ , we have

$$\sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - n j} 2^{-p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}$$

and thus

$$\begin{aligned} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}} &\lesssim [2^{p\gamma_2 j - n j} 2^{-p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}]^{\frac{1}{p}} 2^{-\frac{2j'\beta}{p}} \\ &\lesssim 2^{\gamma_2 j - \frac{n j}{p}} 2^{-j'(\gamma_1 + \frac{n}{2} - \frac{n}{p} + \frac{2\beta}{p})}. \end{aligned}$$



The last estimate implies

$$|b_{j,k}^{\epsilon,1}(t)| \lesssim 2^{\frac{n}{2}+j-\frac{n}{p}+\gamma_2 j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{n(j'-j)(1-\frac{2}{p})} \\ 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p}+\frac{2\beta}{p})} 2^{-2j'\beta(1-\frac{2}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.$$

As a result, we obtain

$$I_{Q_r}^5 = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,1}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right)^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} 2^{pj(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)} e^{-cpt2^{2j\beta}} \right. \\ \left. \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p}-\frac{2\beta}{p}+2\beta)} 2^{n(j'-j)(1-\frac{2}{p})} \right. \right. \\ \left. \left. \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}}.$$

Also, for any  $\delta > 0$ , we have, by  $m' \leq 1$  and  $s2^{2j'\beta} \leq 1$ ,

$$I_{Q_r}^5 \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\ \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p}-\frac{2\beta}{p}+2\beta)} 2^{pn(j'-j)(1-\frac{2}{p})} \times \\ \left. \left[ \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-cpt2^{2j\beta}} (t2^{2j\beta})^m \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) \frac{dt}{t} \right] \right\}^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \left\{ \sum_{j \leq j'+2} 2^{(j'-j)(\delta+pn-2n)} \right. \\ 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p}-\frac{2\beta}{p}+2\beta)} 2^{-2j'\beta} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right) \left. \right\}^{\frac{q}{p}} \\ \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \left\{ \sum_{j \leq j'+2} 2^{(j'-j)(\delta+pn-2n)} \right. \\ 2^{-pj'(2\gamma_2+2-2\beta+n-\frac{2n}{p})} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right) \left. \right\}^{\frac{q}{p}}.$$

Subcase 8.4.1:  $q \leq p$ . Take  $0 < \frac{\delta}{p} < 2\gamma_2 + 2 - 2\beta$  for  $\gamma_1 + \gamma_2 + 1 > 0$ . The  $\alpha$ -triangle inequality implies

$$I_{Q_r}^5 \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j'+2} 2^{q(j-j')(2\gamma_2+2-2\beta-\frac{\delta}{p})} \\ 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{\frac{ds}{s}} \right)^{\frac{q}{p}} \\ \lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.$$

Subcase 8.4.2:  $q > p$ . Take  $0 < \delta < 2\gamma_2 + 2 - 2\beta$ . We use the Hölder inequality to get

$$\begin{aligned} I_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j'+2} 2^{p(j-j')(2\gamma_2+2-2\beta-\delta)} \right)^{\frac{q-p}{p}} \\ &\quad \left\{ \sum_{j \leq j'+2} 2^{p(j-j')(2\gamma_2+2-2\beta-\delta)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\ &\quad \times \left. \left( \int_0^{2^{-1-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m' \frac{ds}{s}} \right)^{\frac{q}{p}} \right\} \\ &\lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

Case 8.5: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 2}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}.$$

For this, we use (4.6) and (ii) of Lemma 4.2 to produce

$$\begin{aligned} &|b_{j, k}^{\epsilon, 2}(t)| \\ &\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{j \leq j'+2} \sum_{w, w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} 2^{n(j'-j)(1-\frac{2}{p})} \\ &\quad \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\ &\lesssim 2^{\frac{nj}{2}+j} e^{-ct2^{2j\beta}} \sum_{w, w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j \leq j'+2} 2^{-\frac{4j'\beta m}{p}} \\ &\quad \times 2^{n(j'-j)(1-\frac{2}{p})} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\ &\quad \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{-\frac{2(m-1)}{p-2}} ds \right)^{\frac{p-2}{p}} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}. \end{aligned}$$

Because  $m > 1$  and  $p > 2$ , one has

$$\left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} s^{-\frac{2(m-1)}{p} \cdot \frac{p}{p-2}} ds \right)^{\frac{p-2}{p}} \lesssim 2^{-2j'\beta} 2^{\frac{4j'\beta m}{p}}.$$

On the other hand, when  $t < r^{2\beta}$ , one has

$$\begin{aligned} v \in \mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III} &\implies \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \\ &\lesssim 2^{p\gamma_2 j - n j} 2^{-p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}. \end{aligned}$$

So, for any  $\delta > 0$ , we obtain

$$\begin{aligned}
I_{Q_r}^6 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,2}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} 2^{p\gamma_2 j-nj} 2^{\frac{npj}{2}+pj} e^{-ct2^{2j\beta}} \right. \\
&\quad \left[ \sum_{j \leq j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{-2j'\beta} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \right. \\
&\quad \left. \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \frac{(s2^{2j'\beta})^m}{s} ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^m \frac{dt}{t} \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-2pj'\beta} 2^{-2pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \\
&\quad \left. \int_{2^{-2j\beta}}^{r^{2\beta}} e^{-cpt2^{2j\beta}} (t2^{2j\beta})^m \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \left\{ \sum_{j \leq j'+2} 2^{(\delta+pn-2n)(j'-j)} \right. \\
&\quad \left. 2^{-pj'(2\gamma_1+n-\frac{2n}{p})} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-1-2j'\beta}}^{\frac{t}{2}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 8.5.1:  $q \leq p$ . Take  $0 < \delta < p(2\gamma_2+2-2\beta)$ . Using the  $\alpha$ -triangle inequality, we get

$$\begin{aligned}
I_{Q_r}^6 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j'+2} 2^{q(j-j')(2\gamma_2+2-2\beta-\frac{\delta}{p})} \\
&\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-1-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

Subcase 8.5.2:  $q > p$ . Take  $0 < \delta < 2\gamma_2+2-2\beta$ . The Hölder's inequality implies

$$\begin{aligned}
I_{Q_r}^6 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j'+2} 2^{p(j-j')(2\gamma_2+2-2\beta-\delta)} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{p(j-j')(2\gamma_2+2-2\beta-\delta)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \left( \int_{2^{-1-2j'\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

Case 8.6: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon,j,k) \in \Lambda_n} b_{j,k}^{\epsilon,3}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p,q,m}^{\gamma_1,\gamma_2,III}.$$

Similarly, by (4.6) and (iii) of Lemma 4.2, we apply Hölder's inequality to get

$$\begin{aligned}
|b_{j,k}^{\epsilon,3}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j \leq j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{\frac{t}{2}}^t \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \right. \\
&\quad \times e^{-c(t-s)2^{2j\beta}} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\
&\lesssim 2^{\frac{nj}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \sum_{j \leq j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds.
\end{aligned}$$

Because of  $v \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, I}$ , one gets

$$\sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \lesssim 2^{p\gamma_2 j - n j} 2^{-p j'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} (s 2^{2j'\beta})^{-m}.$$

So, for  $\frac{t}{2} \leq s \leq t$ , we have

$$\begin{aligned}
|b_{j,k}^{\epsilon,3}(t)| &\lesssim 2^{\frac{nj}{2}+j+\gamma_2 j - \frac{nj}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j'+2} 2^{-j'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \left( \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} (s 2^{2j'\beta})^{-\frac{m}{p}} ds \\
&\lesssim 2^{\frac{nj}{2}+j+\gamma_2 j - \frac{nj}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j \leq j'+2} 2^{-j'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} (t 2^{2j'\beta})^{-\frac{m}{p}} t^{\frac{p-1}{p}}.
\end{aligned}$$

By the above estimate for  $|b_{j,k}^{\epsilon,3}(t)|$ , we have, for any  $\delta > 0$ ,

$$\begin{aligned}
I_{Q_r}^7 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,3}(t)|^p (t2^{2j\beta})^m \frac{dt}{t} \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{qj(\frac{n}{2}+1+\gamma_2-\frac{n}{p})} \\
&\quad \left\{ \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{-j'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \right. \right. \\
&\quad \left. \left. \left( \int_{\frac{t}{2}}^t e^{-c(t-s)2^{2j\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} (t2^{2j'\beta})^{-\frac{m}{p}} \right]^p t^{p-1} (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \\
&\quad \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \int_{2^{-2j\beta}}^{r^{2\beta}} \left( \int_{\frac{t}{2}}^t e^{-cp(t-s)2^{2j\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j'\beta})^{-m} t^{p-1} (t2^{2j\beta})^m \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \\
&\quad \left\{ \sum_{j \leq j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-2pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \int_{2^{-2j\beta}}^{r^{2\beta}} (t2^{2j'\beta})^{-2m} \right. \\
&\quad \left. \left( \int_{\frac{t}{2}}^t e^{-cp(t-s)2^{2j\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) (t2^{2j\beta})^m t^p \frac{dt}{t} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Changing the order of integration, we readily get

$$\int_s^{2s} e^{-cp(t-s)2^{2j\beta}} (t2^{2j'\beta})^{-2m} t^p (t2^{2j\beta})^m \frac{dt}{t} \lesssim 2^{4\beta m(j-j')-2\beta p} 2^{-2pj'\beta}.$$

Hence

$$\begin{aligned}
I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(2\gamma_2+2-2\beta+n-\frac{2n}{p})} \\
&\quad \left[ \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{n(j'-j)(p-2)} 2^{-pj'(2\gamma_2+2-2\beta+n-\frac{2n}{p})} 2^{4\beta m(j-j')} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-1-2(j'+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 8.6.1:  $q \leq p$ . Take  $0 < \frac{\delta}{p} < 2\gamma_2 + 2 - 4\beta + \frac{4\beta m}{p}$ . We use the  $\alpha$ -triangle inequality to get

$$\begin{aligned}
I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j \leq j'+2} 2^{q(j-j')(2\gamma_2+2-4\beta+\frac{4\beta m}{p}-\frac{\delta}{p})} \\
&\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-1-2(j'+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

Subcase 8.6.2:  $q > p$ . Take  $0 < \delta < 2\gamma_2 + 2 - 4\beta + \frac{4\beta m}{p}$ . We use Hölder's inequality to get

$$\begin{aligned} I_{Q_r}^7 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j \leq j' + 2} 2^{p(j-j')(2\gamma_2+2-4\beta+\frac{4\beta m}{p}-\delta)} \right)^{\frac{q-p}{p}} \\ &\quad \left\{ \sum_{j \leq j' + 2} 2^{p(j-j')(2\gamma_2+2-4\beta+\frac{4\beta m}{p}-\delta)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\ &\quad \left. \left( \int_{2^{-1-2(j'+2)\beta}}^{r^{2\beta}} \sum_{(\epsilon', k') \in S_{r^{w, j'}}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\ &\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

## 9. PROOF OF LEMMA 5.4

9.1. **The setting**  $1 < p \leq 2$ . We divide the argument into two cases.

Case 9.1: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 4}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}.$$

Since  $v \in \mathbf{B}_{p,q,m,m'}^{\gamma_1, \gamma_2}$ , one gets

$$|v_{j', k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2} + \gamma_1 - \gamma_2)j'}.$$

By (4.4), Hölder's inequality and (iv) of Lemma 4.2, we get

$$\begin{aligned} &|b_{j,k}^{\epsilon, 4}(t)| \\ &\lesssim 2^{\frac{n}{2}j} \sum_{j < j' + 2} \sum_{\epsilon', k', \epsilon'', k''} \int_0^{2^{-2j'\beta}} \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)| \right. \\ &\quad \times (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\ &\lesssim 2^{\frac{n}{2}j} \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \sum_{\epsilon', k', \epsilon'', k''} \int_0^{2^{-2j'\beta}} \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} \right. \\ &\quad \times (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\ &\lesssim 2^{\frac{n}{2}j} \sum_{j < j' + 2} \sum_{w, w' \in \mathbb{Z}^n} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} (1 + |w|)^{-N} (1 + |w'|)^{-N} \\ &\quad \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j,k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p'}}. \end{aligned}$$

For  $0 < s < 2^{-2j'\beta}$ , the fact  $v \in \mathbf{B}_{p,q}^{\gamma_1, \gamma_2, II}$  implies

$$|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 r}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{Q_{j,k} \subset Q_r} |v_{j,k}^{\epsilon}(t)|^p \right)^{\frac{q}{p}} < \infty.$$

This finiteness amounts to

$$\left( \sum_{Q_{j,k} \subset Q_r} |v_{j,k}^{\epsilon}(t)|^p \right)^{\frac{q}{p}} \lesssim |Q_r|^{\frac{q}{p} - \frac{q\gamma_2}{n}} 2^{-2qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})},$$

that is to say,

$$\sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \lesssim 2^{-nj+p\gamma_2 j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'}.$$

The above estimate yields

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2 j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} 2^{-2j'\beta}.$$

Consequently, we get

$$\begin{aligned} |b_{j,k}^{\epsilon,4}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \left( 2^{-nj+p\gamma_2 j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} 2^{-2j'\beta} \right)^{\frac{p-1}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ &\lesssim 2^{-\frac{nj}{2}+j+\frac{nj}{p}+(p-1)\gamma_2 j} \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \\ &\quad 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{\frac{(p-1)nj'}{p}} 2^{(2-p)\gamma_2 j'} 2^{-\frac{2\beta(p-1)j'}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

whence finding

$$\begin{aligned} II_{Q_r}^4(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,4}(t)|^p \right)^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} 2^{pj[1 + \frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2]} \right. \\ &\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{nj'(1-\frac{1}{p})} 2^{(2-p)\gamma_2 j'} \right. \\ &\quad \left. \times 2^{-2\beta j'(1-\frac{1}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \Big\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{qj[1 + \frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2]} \\ &\quad \left\{ \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{nj'(1-\frac{1}{p})} 2^{(2-p)\gamma_2 j'} \right. \right. \\ &\quad \left. \times 2^{-2\beta j'(1-\frac{1}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \Big\}^{\frac{q}{p}}. \end{aligned}$$

By (4.5), we can see that for  $\delta > 0$ ,

$$\begin{aligned} & \sum_{(\epsilon, k) \in S_{Q_r}^j} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2} + \gamma_1)j'} 2^{nj'(1 - \frac{1}{p})} 2^{(2-p)\gamma_2 j'} \right. \\ & \quad \times \left. 2^{-2\beta j'(1 - \frac{1}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \\ & \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{\delta(j' - j)} 2^{-p(\frac{n}{2} + \gamma_1)j'} 2^{(p-1)n j'} 2^{p(2-p)\gamma_2 j'} \\ & \quad \times 2^{-2\beta j'(p-1)} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) \end{aligned}$$

and

$$\begin{aligned} II_{Q_r}^4(t) & \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + 1 + (p-1)\gamma_2)} \\ & \quad \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{\delta(j' - j)} 2^{-pj'[2\gamma_1 + (p-2)\gamma_2]} 2^{-2\beta(p-1)j'} \right. \\ & \quad \times \left. 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right]. \end{aligned}$$

From  $s2^{2j'\beta} \leq 1$  and  $m' < 1$  it easily follows that

$$2^{-2j'\beta(p-1)} \int_0^{2^{-2j'\beta}} |u_{j', k'}^{\epsilon'}(s)|^p ds \lesssim 2^{-2\beta pj'} \int_0^{2^{-2j'\beta}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s}.$$

Finally, we obtain that for  $\delta > 0$

$$\begin{aligned} II_{Q_r}^4(t) & \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(2-2\beta+p\gamma_2)} \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{\delta(j' - j)} \right. \\ & \quad \times \left. 2^{-pj'(2-2\beta+p\gamma_2)} 2^{pj'[\frac{n}{2} + \gamma_1 - \frac{n}{p}]} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}}. \end{aligned}$$

Subcase 9.1.1:  $q \leq p$ . For this, we obtain, by the  $\alpha$ -triangle inequality,

$$\begin{aligned} II_{Q_r}^4(t) & \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} \sum_{j < j' + 2} 2^{q(j-j')(2-2\beta+p\gamma_2 - \frac{\delta}{p})} \\ & \quad \times 2^{qj'[\frac{n}{2} + \gamma_1 - \frac{n}{p}]} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}}. \end{aligned}$$

If

$$0 < \frac{\delta}{p} < 2(1 - \beta) + p\gamma_2$$

then

$$II_Q^4 \lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}$$

follows from changing the order of  $j$  and  $j'$ .



Subcase 9.1.2:  $q > p$ . By Hölder's inequality, we get

$$\begin{aligned}
II_{Q_r}^4(t) &\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} \left[ \sum_{j < j' + 2} 2^{p(j-j')(2-2\beta+p\gamma_2-\frac{\delta}{p})} \right. \\
&\quad \left. 2^{pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} \frac{|Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}}}{(1+|w|)^N} \sum_{j \geq -\log_2 r} \left( \sum_{j < j' + 2} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} \right)^{\frac{q-p}{p}} \\
&\quad \times \left\{ \sum_{j < j' + 2} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} 2^{qj'(\frac{n}{2}+\gamma_1-\frac{n}{p})} \times \right. \\
&\quad \left. \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbb{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 9.2: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 5}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p,q}^{\gamma_1, \gamma_2, II}.$$

Because  $2^{-2j'\beta} \leq s \leq t$ , one gets

$$|v_{j', k''}^{\epsilon''}(s)| \lesssim 2^{-(\frac{n}{2}+\gamma_1-\gamma_2)j'}.$$

Applying (4.4) and (v) of Lemma 4.2, we get

$$\begin{aligned}
&|b_{j,k}^{\epsilon, 5}(t)| \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \sum_{\epsilon', k', \epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ |u_{j', k'}^{\epsilon'}(s)| |v_{j', k''}^{\epsilon''}(s)|^{p-1} \right. \\
&\quad \left. \times (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \right\} ds \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{j < j' + 2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\
&\quad \int_{2^{-2j'\beta}}^t \left( \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{p-1}{p}} ds \\
&\lesssim 2^{\frac{n}{2}+j} \sum_{j < j' + 2} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\
&\quad \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{p-1}{p}}.
\end{aligned}$$

Because  $v \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$ , we have

$$\begin{aligned}
&(2^{2j'\beta})^{\frac{q}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{q}{p}} \\
&\lesssim \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{-(\frac{q\gamma_2}{n} - \frac{q}{p})} 2^{-qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})}
\end{aligned}$$

and so

$$\int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2 j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j' - 2j'\beta}$$

thanks to  $j \geq -\log_2 r$  and  $p\gamma_2 \leq n$ . Therefore,

$$\begin{aligned} |b_{j,k}^{\epsilon,5}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \left( 2^{(p\gamma_2-n)j} 2^{-p(\gamma_1 + \frac{n}{2} - \frac{n}{p})j'} 2^{-2\beta j'} \right)^{\frac{p-1}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ &\lesssim 2^{-\frac{nj}{2} + \frac{nj}{p} + (p-1)\gamma_2 j} \sum_{j < j'+2} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} 2^{-(\frac{n}{2}+\gamma_1)j'} \\ &\quad 2^{\frac{(p-1)nj'}{p}} 2^{(2-p)\gamma_2 j'} 2^{-\frac{2\beta(p-1)j'}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

The above estimate implies

$$\begin{aligned} II_{Q_r}^5(t) &= |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \sum_{(\epsilon, k) \in S_r^j} |b_{j,k}^{\epsilon,5}(t)|^p \right)^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \sum_{(\epsilon, k) \in S_r^j} 2^{pj[1 + \frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2]} \right. \\ &\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j' + \frac{(p-1)nj'}{p} + (2-p)\gamma_2 j'} \right. \\ &\quad \left. \left. 2^{-\frac{2\beta(p-1)j'}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{qj[1 + \frac{n}{p} - \frac{n}{2} + (p-1)\gamma_2]} \\ &\quad \left\{ \sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j' + \frac{(p-1)nj'}{p} + (2-p)\gamma_2 j'} \right. \right. \\ &\quad \left. \left. 2^{-\frac{2\beta(p-1)j'}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right\}^{\frac{q}{p}}. \end{aligned}$$

By (4.5), we have

$$\begin{aligned} &\sum_{(\epsilon, k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{\frac{(p-1)nj'}{p}} 2^{(2-p)\gamma_2 j'} \right. \\ &\quad \left. 2^{-\frac{2\beta(p-1)j'}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \\ &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(\frac{n}{2}+\gamma_1)j' + (p-1)nj' + p(2-p)\gamma_2 j'} \\ &\quad 2^{-2\beta(p-1)j'} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right). \end{aligned}$$

By a simple computation, we can get

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(2\gamma_1+(p-2)\gamma_2)j'} 2^{-2\beta(p-1)j'} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right].
\end{aligned}$$

Because  $m > 1$  and  $2^{-2j'\beta} \leq s \leq t$ , it is easy to get

$$\begin{aligned}
&2^{-2\beta(p-1)j'} \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \\
&\lesssim 2^{-2\beta pj'} \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2\beta j'})^m \frac{ds}{s}.
\end{aligned}$$

The term  $II_Q^5$  can be estimated as

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-pj'(2-2\beta+p\gamma_2)} 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2\beta j'})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right].
\end{aligned}$$

Subcase 9.2.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have, by  $0 < \frac{\delta}{p} < 2(1-\beta) + p\gamma_2$ ,

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} \\
&\quad \sum_{j < j'+2} 2^{q\delta(j'-j)/p} 2^{-qj'(2-2\beta+p\gamma_2)} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \times \\
&\quad \left[ \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2\beta j'})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}},
\end{aligned}$$

where we have changed the order of  $j$  and  $j'$ .

Subcase 9.2.2:  $q > p$ . Similarly, we use the Hölder inequality to derive

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r < j < -\frac{\log_2 t}{2\beta}} \left\{ \sum_{j < j' + 2} 2^{p(j-j')(2-2\beta+p\gamma_2-\frac{\delta}{p})} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2\beta j'})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \left( \sum_{j < j' + 2} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} \right)^{\frac{q-p}{p}} \\
&\quad \sum_{j \geq -\log_2 r} \left\{ \sum_{j < j'} 2^{(j-j')[2p(1-\beta)+p^2\gamma_2-\delta]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2\beta j'})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbb{B}_{p, q, m}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$

9.2. **The setting  $2 < p < \infty$ .** . We still divide the proof into two cases.

Case 9.3. Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbb{B}_{p, q}^{\gamma_1, \gamma_2, II}.$$

For this, we have, by Hölder's inequality, (4.6) and (iv) of Lemma 4.2,

$$\begin{aligned}
|b_{j, k}^{\epsilon, 4}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j' + 2} \sum_{w, w' \in \mathbb{Z}^n} \left( \frac{1}{(1+|w|)^N} \right) \left( \frac{1}{(1+|w-w'|)^N} \right) 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \int_0^{2^{-2j'\beta}} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
&\lesssim 2^{\frac{nj}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} \left( \frac{1}{(1+|w|)^N} \right) \left( \frac{1}{(1+|w-w'|)^N} \right) \sum_{j < j' + 2} 2^{-2j'\beta(1-\frac{2}{p})} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Because  $v \in \mathbb{B}_{p, q}^{\gamma_1, \gamma_2, II}$ , one has

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2j} 2^{-p(\gamma_1+\frac{n}{2}-\frac{n}{p})j'-2\beta j'}.$$

This in turn implies

$$\begin{aligned}
|b_{j, k}^{\epsilon, 4}(t)| &\lesssim 2^{\frac{nj}{2}+j-\frac{nj}{p}+\gamma_2j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} 2^{-\frac{2j'\beta(p-2)}{p}} \\
&\quad 2^{n(j'-j)(1-\frac{2}{p})} 2^{-\frac{2\beta j'}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

The above estimate for  $|b_{j,k}^{\epsilon,4}(t)|$  derives

$$\begin{aligned}
II_{Q_r}^4(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,4}(t)|^p \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)qj} \\
&\quad \left\{ \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(\gamma_1+\frac{n}{2}-\frac{n}{p})j'} 2^{-2j' \frac{\beta(p-1)}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \right. \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{jk}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj[(p-2)n+p(\gamma_1+1+\gamma_2)]/p} \\
&\quad \left\{ \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} 2^{-\frac{2j'\beta(p-1)}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \right. \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{jk}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \Big\}^{\frac{q}{p}}.
\end{aligned}$$

Since  $0 < s < 2^{-2j'\beta}$ , we get for any  $\delta > 0$ ,

$$\begin{aligned}
&\sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} 2^{-\frac{2j'\beta(p-1)}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{jk}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \\
&\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})pj'} 2^{-2j'\beta p} 2^{(\delta+np-2n)(j'-j)} \\
&\quad \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right)
\end{aligned}$$

and

$$\begin{aligned}
II_{Q_r}^4(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(n-\frac{2n}{p}+\gamma_1+1+\gamma_2)} \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \sum_{j < j'+2} 2^{-pj'(\frac{n}{2}+\gamma_1-\frac{n}{p}+2\beta)} 2^{(\delta+np-2n)(j'-j)} \\
&\quad \left. \times \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right) \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 9.3.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned}
& II_{Q_r}^4(t) \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{-\log_2 r \leq j < -\frac{\log_2 t}{2\beta}} 2^{qj(n - \frac{2n}{p} + \gamma_1 + 1 + \gamma_2)} \\
& \quad \sum_{j < j' + 2} \left[ 2^{-(\frac{n}{2} + \gamma_1 - \frac{n}{p} + 2\beta)qj'} 2^{q(\frac{\delta}{p} + n - \frac{2n}{p})(j' - j)} \right. \\
& \quad \quad \left. \times \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m' \frac{ds}{s}} \right)^{\frac{q}{p}} \right] \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j' + 2} 2^{q(j - j')[n(p - 2) + p(\gamma_1 + 1 + \gamma_2)]/p} \\
& \quad 2^{q(\frac{\delta}{p} + n - \frac{2n}{p})(j' - j)} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m' \frac{ds}{s}} \right)^{\frac{q}{p}}.
\end{aligned}$$

If  $\gamma_1 + \gamma_2 + 1 > 0$ , then selecting  $0 < \delta < p(\gamma_1 + \gamma_2 + 1)$  we get

$$II_{Q_r}^4(t) \lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.$$

Subcase 9.3.2:  $q > p$ . Take  $0 < \delta < (\gamma_1 + \gamma_2 + 1)$ . By Hölder's inequality, we have

$$\begin{aligned}
& II_{Q_r}^4(t) \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left\{ 2^{j[(p - 2)n + p(\gamma_1 + 1 + \gamma_2)]} \sum_{j < j' + 2} 2^{pj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\
& \quad \left. 2^{-j'[n(p - 2) + p(\gamma_1 + \gamma_2 + 1)]} 2^{(\delta + pn - 2n)(j' - j)} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m' \frac{ds}{s}} \right\}^{\frac{q}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left( \sum_{j < j' + 2} 2^{p(j - j')(\gamma_1 + 1 + \gamma_2 - \frac{\delta}{p})} \right)^{\frac{q - p}{p}} \\
& \quad \times \left\{ \sum_{j < j' + 2} 2^{p(j - j')(\gamma_1 + 1 + \gamma_1 - \frac{\delta}{p})} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m' \frac{ds}{s}} \right)^{\frac{q}{p}} \right\} \\
& \lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$

Case 9.4: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 5}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q}^{\gamma_1, \gamma_2, II}.$$

By (4.6) and (v) of Lemma 4.2, we have

$$\begin{aligned}
|b_{j,k}^{\epsilon,5}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{j < j'+2} \sum_{\epsilon', k', \epsilon'', k''} \int_{2^{-2j'\beta}}^t \left\{ |u_{j',k'}^{\epsilon'}(s)| |v_{j',k''}^{\epsilon''}(s)| \right. \\
&\quad \times (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k' - k''|)^{-N} \Big\} ds \\
&\lesssim 2^{\frac{n_j}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w-w'|)^N} \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \int_{2^{-2j'\beta}}^t \left( \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
&\lesssim 2^{\frac{n_j}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w-w'|)^N} \sum_{j < j'+2} 2^{-\frac{4m\beta j'}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \left( \int_{2^{-2j'\beta}}^t s^{\frac{2-2m}{p-2}} ds \right)^{\frac{p-2}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

On the one hand, it is easy to establish

$$\left( \int_{2^{-2j'\beta}}^t s^{\frac{2-2m}{p-2}} ds \right)^{\frac{1}{u}} \lesssim 2^{-2j'\beta(p-2m)/p}.$$

On the other hand,  $v(t, x) \in \mathbb{B}_{p,q,m}^{\gamma_1, \gamma_2, III}$  implies

$$\int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \lesssim 2^{-nj+pj\gamma_2} 2^{-pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})}.$$

So, we can get

$$\begin{aligned}
|b_{j,k}^{\epsilon,5}(t)| &\lesssim 2^{\frac{n_j}{2}+j} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \\
&\quad \sum_{j < j'+2} 2^{-2j'\beta} 2^{n(j'-j)(1-\frac{2}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{jk}^{w', j'}} |v_{j',k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
&\lesssim 2^{j(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-2j'\beta} 2^{n(j'-j)(1-\frac{2}{p})} \\
&\quad \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{jk}^{w, j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

Upon letting  $0 < \delta < p(\gamma_1 + 1 + \gamma_2)$ , we use the above estimate for  $|b_{j,k}^{\epsilon,5}(t)|$  to produce

$$\begin{aligned}
II_{Q_r}^5(t) &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,5}(t)|^p \right)^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \sum_{(\epsilon,k) \in S_r^j} 2^{pj(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)} \right. \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-2j'\beta} 2^{n(j'-j)(1-\frac{2}{p})} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \right]^p \Bigg\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{qj(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)} \\
&\quad \left\{ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{(\delta+pn-2n)(j'-j)} 2^{-pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})-2pj'\beta} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 9.4.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality and  $0 < \delta < p(\gamma_1 + 1 + \gamma_2)$ , we obtain

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j'+2} 2^{q(j-j')(\gamma_1+1+\gamma_2-\frac{\delta}{p})} \\
&\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

Subcase 9.4.2:  $q > p$ . Similarly, we have

$$\begin{aligned}
II_{Q_r}^5(t) &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} \left\{ \sum_{j < j'+2} 2^{p(j-j')(\gamma_1+1+\gamma_2-\frac{\delta}{p})} \right. \\
&\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right) \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} \left( \sum_{j < j'+2} 2^{(j-j')[p(\gamma_1+1+\gamma_2)-\delta]} \right)^{\frac{q-p}{p}} \\
&\quad \left\{ \sum_{j < j'+2} 2^{(j-j')[p(\gamma_1+1+\gamma_2)-\delta]} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1,\gamma_2,III}} + \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1,\gamma_2,IV}}.
\end{aligned}$$

## 10. PROOF OF LEMMA 5.5

10.1. **The setting**  $1 < p \leq 2$ . We divide the proof into four cases.



Case 10.1: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}.$$

Because  $0 < s < 2^{-2j'\beta}$ , one gets

$$|v_{j', k''}^{\epsilon''}(s)| \lesssim 2^{-\frac{nj'}{2}} 2^{-j'(\gamma_1 - \gamma_2)}.$$

We use (4.4) and (iv) of Lemma 4.2 to get

$$\begin{aligned} & |b_{j, k}^{\epsilon, 4}(t)| \\ & \lesssim 2^{\frac{nj}{2} + j} \sum_{j < j' + 2} \sum_{\epsilon', k', \epsilon'', k''} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \int_0^{2^{-2j'\beta}} \{ |u_{j', k'}^{\epsilon'}(s)| \\ & \quad \times |v_{j', k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'} k' - k|)^{-N} (1 + |k'' - k'|)^{-N} \} ds \\ & \lesssim 2^{\frac{nj}{2} + j} \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \\ & \quad \int_0^{2^{-2j'\beta}} \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{p-1}{p}} ds \\ & \lesssim 2^{\frac{nj}{2} + j} \sum_{j < j' + 2} 2^{-(2-p)(\frac{n}{2} + \gamma_1 - \gamma_2)j'} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w'|)^{-N} \\ & \quad \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \right)^{\frac{p-1}{p}}. \end{aligned}$$

By the following estimate

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj + p\gamma_2 j} 2^{-p(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'} 2^{-2j'\beta},$$

we have

$$\begin{aligned} |b_{j, k}^{\epsilon, 4}(t)| & \lesssim 2^{-\frac{nj}{2} + j + \frac{nj}{p} + (p-1)\gamma_2 j} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2} + \gamma_1)j'} \\ & \quad 2^{\frac{nj'(p-1)}{p} + (2-p)\gamma_2 j' - \frac{2j'\beta(p-1)}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

For  $II^4(u, v)$  and  $t2^{2j\beta} < 1$ , we then get

$$\begin{aligned} II_{Q_r}^4 & = |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} |b_{j, k}^{\epsilon, 4}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\ & \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon, k) \in S_r^j} 2^{-\frac{pnj}{2} + pj + nj + p(p-1)\gamma_2 j} \right. \\ & \quad \left[ \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \sum_{j < j' + 2} 2^{-(\frac{n}{2} + \gamma_1)j'} 2^{\frac{nj'(p-1)}{p} + (2-p)\gamma_2 j'} \right. \\ & \quad \left. \left. 2^{-\frac{2j'\beta(p-1)}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right\}^{\frac{q}{p}}. \end{aligned}$$

Set

$$A_{j,k} = \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{\frac{nj'(p-1)}{p}+(2-p)\gamma_2 j'} \right. \\ \left. 2^{-\frac{2j'\beta(p-1)}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p.$$

Applying Hölder's inequality to  $w$  yields

$$A_{j,k} \lesssim \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{\frac{(p-1)nj'}{p}+(2-p)\gamma_2 j' - \frac{2\beta(p-1)j'}{p}} \right. \\ \left. \left( \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p.$$

Hence we apply (4.5) to get that for  $\delta > 0$ ,

$$\begin{aligned} II_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} \\ &\quad \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \left[ \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'} 2^{\frac{(p-1)nj'}{p}+(2-p)\gamma_2 j' - \frac{2\beta(p-1)j'}{p}} \right. \right. \\ &\quad \left. \left. \left( \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j'\beta})^{m' \frac{dt}{t}} \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \right. \\ &\quad \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-p(\frac{n}{2}+\gamma_1)j'} 2^{(p-1)nj'+p(2-p)\gamma_2 j' - 2\beta(p-1)j'} \\ &\quad \left. \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j'\beta})^{m' \frac{dt}{t}} \right\}^{\frac{q}{p}} \\ &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} 2^{-\frac{q\delta j}{p}} \\ &\quad \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \sum_{j < j'+2} 2^{\delta j'} 2^{-p(\frac{n}{2}+\gamma_1)j'} 2^{(p-1)nj'+p(2-p)\gamma_2 j' - 2\beta(p-1)j'} \right. \\ &\quad \left. \sum_{w \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j'\beta})^{m' \frac{dt}{t}} \right\}^{\frac{q}{p}}. \end{aligned}$$

Because  $\int_0^{2^{-2j\beta}} (t2^{2j\beta})^m \frac{dt}{t} \lesssim 1$ , we obtain

$$\begin{aligned} II_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} 2^{-\frac{q\delta j}{p}} \\ &\quad \left[ \sum_{j < j'+2} 2^{\delta j'} 2^{p j'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{2\beta j'} 2^{-p j'(2-2\beta+p\gamma_2)} \right. \\ &\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) \right]^{\frac{q}{p}}. \end{aligned}$$

Subcase 10.1.1:  $q \leq p$ . Notice that  $p\gamma_2 + 2 - 2\beta > 0$ . So, upon taking

$$0 < \delta < p(\gamma_1 + 1 + (p-1)\gamma_2)$$

we obtain

$$\begin{aligned} II_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+1+(p-1)\gamma_2)} 2^{-\frac{q\delta j}{p}} \\ &\quad \left[ \sum_{j < j'+2} 2^{q\delta j'/p} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{2q\beta j'/p} 2^{-qj'[2(1-\beta)+p\gamma_2]} \right. \\ &\quad \left. \times \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right] \\ &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j'+2} 2^{q(\frac{\delta}{p}-p\gamma_2-2+2\beta)(j'-j)} \\ &\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{\frac{2q\beta j'}{p}} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right]^{\frac{q}{p}} \\ &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j'+2} 2^{q(\frac{\delta}{p}-p\gamma_2-2+2\beta)(j'-j)} \\ &\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}}, \end{aligned}$$

where we have used the fact that

$$(s2^{2j'\beta})^{1-m'} \leq 1 \text{ for } s \leq 2^{-2j'\beta} \text{ \& } 1-m' > 0.$$

Changing the order of  $j$  and  $j'$  yields  $II_Q^4 \lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}$ .

Subcase 10.1.2:  $q > p$ . We have, by Hölder's inequality,

$$\begin{aligned} II_{Q_r}^4 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} \left[ \sum_{j < j'+2} 2^{p(2-2\beta+p\gamma_2-\frac{\delta}{p})(j-j')} \right. \\ &\quad \left. 2^{pj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{2\beta j'} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right] \\ &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-N} \sum_{j \geq -\log_2 r} \left( \sum_{j < j'+2} 2^{p(2-2\beta+p\gamma_2-\frac{\delta}{p})(j-j')} \right)^{\frac{q-p}{p}} \\ &\quad \left\{ \sum_{j < j'+2} 2^{[2p(1-\beta)+p^2\gamma_2-\delta](j-j')} 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \right. \\ &\quad \left. \times \left( 2^{2\beta j'} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right\} \\ &\lesssim \|u\|_{\mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}}. \end{aligned}$$

Case 10.2: Under  $1 < p \leq 2$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j,k}^{\epsilon, 5}(t) \Phi_{j,k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p,q,m'}^{\gamma_1, \gamma_2, IV}.$$

For  $2^{-2j'\beta} \leq s \leq t$  and  $0 < j' - j'' \leq 3$ , we have

$$|v_{j'',k''}^{\epsilon''}(s)| \lesssim 2^{-\frac{nj''}{2}} 2^{(\gamma_2 - \gamma_1)j'}.$$

The inequality (4.4) and (v) of Lemma 4.2 implies

$$\begin{aligned} |b_{j,k}^{\epsilon,5}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j'+2} \sum_{\epsilon',k',\epsilon'',k''} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \int_{2^{-2j'\beta}}^t \left\{ |u_{j',k'}^{\epsilon'}(s)| \right. \\ &\quad \times |v_{j'',k''}^{\epsilon''}(s)|^{p-1} (1 + |2^{j-j'}k' - k|)^{-N} (1 + |k'' - k'|)^{-N} \Big\} ds \\ &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j'+2} \sum_{w,w' \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w'|)^N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \int_{2^{-2j'\beta}}^t \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j'',k''}^{\epsilon''}(s)|^p \right)^{\frac{p-1}{p}} ds \\ &\lesssim 2^{\frac{nj}{2}+j} \sum_{j < j'+2} \sum_{w,w' \in \mathbb{Z}^n} \frac{1}{(1+|w|)^N (1+|w'|)^N} 2^{-(2-p)(\frac{n}{2}+\gamma_1-\gamma_2)j'} \\ &\quad \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j'',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{p-1}{p}}. \end{aligned}$$

Because  $2^{-2j'\beta} \leq s \leq t$ , we have  $|v_{j'',k''}^{\epsilon''}(s)| \lesssim 2^{-\frac{nj''}{2}} 2^{j'(\gamma_2 - \gamma_1)}$ , thereby getting

$$\int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |v_{j'',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-p(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} 2^{-2j'\beta} 2^{pj(\gamma_2 - \frac{n}{p})}.$$

The above estimate implies

$$\begin{aligned} |b_{j,k}^{\epsilon,5}(t)| &\lesssim 2^{\frac{nj}{2}+j} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(2-p)(\gamma_1-\gamma_2+\frac{n}{2})j'} 2^{-(p-1)(\frac{n}{2}+\gamma_1-\frac{n}{p})j'} \\ &\quad 2^{-\frac{2j'\beta(p-1)}{p}} 2^{j(p-1)(\gamma_2-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \\ &\lesssim 2^{-\frac{nj}{2}+j+\frac{nj}{p}+(p-1)\gamma_2 j} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'+\frac{(p-1)nj'}{p}} \\ &\quad 2^{(2-p)\gamma_2 j'} 2^{-\frac{2j'\beta(p-1)}{p}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

This in turn yields

$$\begin{aligned}
II_{Q_r}^5 &= |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left[ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,5}(t)|^p (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \left[ 2^{-\frac{n}{2}j+j+\frac{n}{p}j+(p-1)\gamma_2j} \right. \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'+\frac{(p-1)n}{p}j'} 2^{(2-p)\gamma_2j'} 2^{-\frac{2j'\beta(p-1)}{p}} \right. \\
&\quad \left. \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^{m'} \frac{dt}{t} \Big\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} 2^{-\frac{pn}{2}j+pj+nj+p(p-1)\gamma_2j} \right. \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1)j'+\frac{(p-1)n}{p}j'} 2^{(2-p)\gamma_2j'} 2^{-\frac{2j'\beta(p-1)}{p}} \right. \\
&\quad \left. \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j\beta})^{m'} \frac{dt}{t} \Big\}^{\frac{q}{p}}.
\end{aligned}$$

It is easy to see that

$$\int_0^{2^{-2j\beta}} (t2^{2j\beta})^{m'} \frac{dt}{t} \lesssim 1.$$

Similarly, by (4.5), we can get that for  $\delta > 0$

$$\begin{aligned}
II_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj[\gamma_1+1+(p-1)\gamma_2]} \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{(\epsilon,k) \in S_r^j} \right. \\
&\quad \sum_{j < j'+2} 2^{\delta(j'-j)} 2^{-pj'(\frac{n}{2}+\gamma_1)} 2^{(p-1)nj'} 2^{p(2-p)\gamma_2j'} 2^{-2\beta(p-1)j'} \\
&\quad \left. \int_0^{2^{-2j'\beta}} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) (t2^{2j\beta})^{m'} \frac{dt}{t} \right]^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj[\gamma_1+1+(p-1)\gamma_2]} 2^{-\frac{q\delta j}{p}} \\
&\quad \left[ \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{\delta j'} 2^{-pj'[2(1-\beta)+p\gamma_2]} 2^{2\beta j'} \right. \\
&\quad \left. 2^{pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon',k') \in S_r^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right) \right]^{\frac{q}{p}}.
\end{aligned}$$

Subcase 10.2.1:  $q \leq p$ . For  $m > 1$  and  $2^{-2j'\beta} \leq s \leq t$ , one has

$$\int_{2^{-2j'\beta}}^t |u_{j',k'}^{\epsilon'}(s)|^p 2^{2\beta j'} ds \lesssim \int_{2^{-2j'\beta}}^t |u_{j',k'}^{\epsilon'}(s)|^p (s2^{2j'\beta})^m \frac{ds}{s}.$$

Notice also that  $p\gamma_2 + 2 - 2\beta > 0$ . So, we have, via taking  $0 < \delta < p(\gamma_1 + 1 + (p-1)\gamma_2)$ ,

$$\begin{aligned}
& II_{Q_r}^5 \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj[\gamma_1 + 1 + (p-1)\gamma_2]} 2^{-\frac{q\delta j}{p}} \\
& \quad \sum_{j < j' + 2} 2^{\frac{q\delta j'}{p}} 2^{-qj'(2-2\beta+p\gamma_2)} 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left[ 2^{2\beta j'} \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right]^{\frac{q}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj[\gamma_1 + 1 + (p-1)\gamma_2]} 2^{-\frac{q\delta j}{p}} \\
& \quad \sum_{j < j' + 2} 2^{qj'(\frac{\delta}{p} - 2 + 2\beta - p\gamma_2)} 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-\frac{qN}{p}} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j' + 2} 2^{q(\frac{\delta}{p} - p\gamma_2 + 2\beta - 2)(j' - j)} \\
& \quad 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left[ \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right]^{\frac{q}{p}} \\
& \lesssim \|u\|_{\mathbf{B}_{p, q, m}^{\gamma_1, \gamma_2, III}},
\end{aligned}$$

where we have changed the order of  $j$  and  $j'$ .

Subcase 10.2.2:  $q > p$ . By using Hölder's inequality and changing the order of  $j$  and  $j'$ , we have

$$\begin{aligned}
& II_{Q_r}^5 \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + 1 + (p-1)\gamma_2) - \frac{q\delta j}{p}} \\
& \quad \left[ \sum_{j < j' + 2} 2^{\delta j' - p[2(1-\beta) + p\gamma_2]j' + 2\beta j'} 2^{p(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) \right]^{\frac{q}{p}} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left[ 2^{pj(\gamma_1 + 1 + (p-1)\gamma_2) - \delta j} \right. \\
& \quad \sum_{j < j' + 2} 2^{\delta j' - p(2-2\beta + p\gamma_2)j'} 2^{2\beta j'} 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right) \left. \right]^{\frac{q}{p}} \\
& \lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left\{ \sum_{j < j' + 2} 2^{[2p(1-\beta) + p^2\gamma_2 - \delta](j-j')} 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \right. \\
& \quad \times \left[ \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right]^{\frac{q}{p}} \left. \right\} \\
& \lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

10.2. **The setting**  $2 < p < \infty$ . We divide the proof into two cases.

Case 10.3: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 4}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}.$$

By (4.6) and (iv) of Lemma 4.2, we have

$$\begin{aligned}
& |b_{j,k}^{\epsilon,4}(t)| \\
& \lesssim 2^{\frac{nj}{2}+j} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j < j'+2} 2^{n(j'-j)(1-\frac{2}{p})} \\
& \int_0^{2^{-2j'\beta}} \left( \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
& \lesssim 2^{\frac{nj}{2}+j} \sum_{w,w' \in \mathbb{Z}^n} (1+|w|)^{-N} (1+|w-w'|)^{-N} \sum_{j < j'+2} 2^{-\frac{2j'\beta(p-2)}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \\
& \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\int_0^{2^{-2j'\beta}} \sum_{(\epsilon'',k'') \in S_{j,k}^{w',j'}} |v_{j',k''}^{\epsilon''}(s)|^p ds \lesssim 2^{-nj+p\gamma_2j} 2^{-p(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-2\beta j'},$$

we have

$$\begin{aligned}
|b_{j,k}^{\epsilon,4}(t)| & \lesssim 2^{\frac{nj}{2}+j+\gamma_2j-\frac{nj}{p}} \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-\frac{2\beta j'(p-2)}{p}} 2^{-\frac{2\beta j'}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \\
& \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

This last estimate is used to give

$$\begin{aligned}
II_{Q_r}^4 & = |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} |b_{j,k}^{\epsilon,4}(t)|^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right)^{\frac{q}{p}} \\
& \lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} \sum_{(\epsilon,k) \in S_r^j} \left[ 2^{\frac{nj}{2}+j+\gamma_2j-\frac{nj}{p}} \right. \right. \\
& \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-\frac{2\beta j'(p-2)}{p}} 2^{-\frac{2\beta j'}{p}} 2^{n(j'-j)(1-\frac{2}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \\
& \left. \left. \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon',k') \in S_{j,k}^{w,j'}} |u_{j',k'}^{\epsilon'}(s)|^p ds \right)^{\frac{1}{p}} \right]^p (t2^{2j'\beta})^{m'} \frac{dt}{t} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 10.3.1:  $q \leq p$ . By the  $\alpha$ -triangle inequality, we have

$$\begin{aligned}
II_{Q_r}^4 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} 2^{qj(\frac{n}{2} + 1 + \gamma_2 - \frac{n}{p})} \\
&\quad \left\{ \sum_{j < j' + 2} 2^{q(\frac{\delta}{p} + n - \frac{2n}{p})(j' - j)} 2^{-q(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'} 2^{-\frac{2\beta(p-1)qj'}{p}} \right. \\
&\quad \left. \times \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right)^{\frac{q}{p}} \right\} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(n - \frac{2n}{p} + 2\gamma_2 + 2 - 2\beta)} \\
&\quad \left\{ \sum_{j < j' + 2} 2^{q(\frac{\delta}{p} + n - \frac{2n}{p})(j' - j)} 2^{-2qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} 2^{-2q\beta j'} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\
&\quad \left. \times \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \right\}.
\end{aligned}$$

Taking  $0 < \delta < 2p\gamma_2 + 2(1 - \beta)p$  and changing the order of  $j$  and  $j'$ , we read off

$$\begin{aligned}
II_{Q_r}^4 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j' + 2} 2^{q(j - j')(2\gamma_2 + 2 - 2\beta - \frac{\delta}{p})} \\
&\quad 2^{qj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbb{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Subcase 10.3.2:  $q > p$ . By Hölder's inequality, we have

$$\begin{aligned}
II_{Q_r}^4 &\lesssim |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} \sum_{j \geq -\log_2 r} \left\{ 2^{pj(\frac{n}{2} + 1 + \gamma_2 - \frac{n}{p})} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \sum_{j < j' + 2} 2^{(\delta + pn - 2n)(j' - j)} \right. \\
&\quad \left. 2^{-p(\frac{n}{2} + \gamma_1 - \frac{n}{p})j' - 2\beta(p-1)j'} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1 + |w|)^N} \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p ds \right\}^{\frac{q}{p}} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left\{ 2^{pj(\frac{n}{2} + 1 + \gamma_2 - \frac{n}{p})} 2^{pj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\
&\quad \left. \sum_{j < j' + 2} 2^{(\delta + pn - 2n)(j' - j)} 2^{-pj'(\frac{n}{2} + \gamma_2 - \frac{n}{p} + 1 - \frac{2\beta}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \frac{ds}{s} \right)^{\frac{q}{p}} \right\}.
\end{aligned}$$



Upon letting  $0 < \delta < p(\gamma_1 + \gamma_2 + 1)$ , we get

$$\begin{aligned}
& II_{Q_r}^4 \\
& \lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left\{ \sum_{j < j' + 2} 2^{(j-j')[2p\gamma_2 + 2(1-\beta)p - \delta]} \right. \\
& \quad \left. 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
& \lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left( \sum_{j < j' + 2} 2^{p(j-j')(2\gamma_2 + 2 - 2\beta - \frac{\delta}{p})} \right)^{\frac{q-p}{p}} \\
& \quad \times \left\{ \sum_{j < j' + 2} 2^{p(j-j')(2\gamma_2 + 2 - 2\beta - \frac{\delta}{p})} 2^{qj'(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \right. \\
& \quad \left. \left( \int_0^{2^{-2j'\beta}} \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^{m'} \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
& \lesssim \|u\|_{\mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}}.
\end{aligned}$$

Case 10. 4: Under  $2 < p < \infty$ ,

$$(t, x) \mapsto \sum_{(\epsilon, j, k) \in \Lambda_n} b_{j, k}^{\epsilon, 5}(t) \Phi_{j, k}^{\epsilon}(x) \text{ is in } \mathbf{B}_{p, q, m'}^{\gamma_1, \gamma_2, IV}.$$

(4.6) and (v) of Lemma 4.2 are employed to produce

$$\begin{aligned}
& |b_{j, k}^{\epsilon, 5}(t)| \\
& \lesssim 2^{\frac{nj}{2} + j} \sum_{w, w' \in \mathbb{Z}^n} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j < j' + 2} 2^{n(j'-j)(1 - \frac{2}{p})} \\
& \quad \int_{2^{-2j'\beta}}^t \left( \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p \right)^{\frac{1}{p}} \left( \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p \right)^{\frac{1}{p}} ds \\
& \lesssim \sum_{w, w' \in \mathbb{Z}^n} 2^{\frac{nj}{2} + j} (1 + |w|)^{-N} (1 + |w - w'|)^{-N} \sum_{j < j' + 2} 2^{-2j'\beta} 2^{n(j'-j)(1 - \frac{2}{p})} \\
& \quad \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{j, k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

Meanwhile, it is easy to deduce that

$$\int_{2^{-2j'\beta}}^t \sum_{(\epsilon'', k'') \in S_{j, k}^{w', j'}} |v_{j', k''}^{\epsilon''}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \lesssim 2^{jp(\gamma_2 - \frac{n}{p})} 2^{-p(\frac{n}{2} + \gamma_1 - \frac{n}{p})j'}.$$

So, we can readily get

$$\begin{aligned}
II_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j'\beta}} \sum_{(\epsilon, k) \in S_r^j} \left[ 2^{\frac{nj}{2}+j-\frac{n}{p}+\gamma_2 j} \right. \right. \\
&\quad \left. \sum_{j < j'+2} 2^{-(\frac{n}{2}+\gamma_1-\frac{n}{p})j'-2j'\beta} 2^{n(j'-j)(1-\frac{2}{p})} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \left. \left. \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_{j,k}^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{1}{p}} \right]^p (t 2^{2j\beta})^{m'} \frac{dt}{t} \right\}^{\frac{q}{p}} \\
&\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ \int_0^{2^{-2j\beta}} 2^{\frac{pnj}{2}+pj-nj+p\gamma_2 j} \right. \\
&\quad \left. \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \sum_{j < j'+2} 2^{(\delta+pn-2n)(j'-j)} 2^{-pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})-2j'p\beta} \right. \\
&\quad \left. \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} (t 2^{2j\beta})^{m'} \frac{dt}{t} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Because  $\int_0^{2^{-2j\beta}} (t 2^{2j\beta})^{m'} \frac{dt}{t} \lesssim 1$ , we obtain

$$\begin{aligned}
II_{Q_r}^5 &\lesssim |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left\{ 2^{\frac{pnj}{2}+pj-nj+p\gamma_2 j} \sum_{w \in \mathbb{Z}^n} (1+|w|)^{-N} \right. \\
&\quad \left. \sum_{j < j'+2} 2^{(\delta+pn-2n)(j'-j)} 2^{-pj'(\frac{n}{2}+\gamma_1-\frac{n}{p})-2j'p\beta} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\}^{\frac{q}{p}}.
\end{aligned}$$

Subcase 10.4.1:  $q \leq p$ . Putting  $0 < \delta < p(\gamma_1 + \gamma_2 + 1)$ , we find

$$\begin{aligned}
II_{Q_r}^5 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1+\frac{n}{2}-\frac{n}{p})} 2^{qj(\frac{n}{2}+1-\frac{n}{p}+\gamma_2)} \\
&\quad \left\{ \sum_{j < j'+2} 2^{q(\frac{\delta}{p}+n-\frac{2n}{p})(j'-j)} 2^{-qj'(\frac{n}{2}+\gamma_1-\frac{n}{p})-2j'q\beta} \right. \\
&\quad \left. \times \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n}-\frac{q}{p}} (1+|w|)^{-\frac{qN}{p}} \sum_{j \geq -\log_2 r} \sum_{j < j'+2} 2^{(j-j')(\gamma_1+\gamma_2+1-\frac{\delta}{p})} \\
&\quad 2^{qj'(\gamma_1+\frac{n}{2}-\frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$

Subcase 10.4.2:  $q > p$ . By Hölder's inequality, we have

$$\begin{aligned}
II_{Q_r}^5 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} 2^{qj(\gamma_1 + \frac{n}{2} - \frac{n}{p})} \\
&\quad \left\{ 2^{pj(\frac{n}{2} + 1 - \frac{n}{p} + \gamma_2)} \sum_{j < j' + 2} 2^{(\delta + pn - 2n)(j' - j)} 2^{-pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p}) - 2j'p\beta} \right. \\
&\quad \times \left. \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&= \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left\{ 2^{pj(\gamma_1 + \gamma_2 + 1 + n - \frac{2n}{p})} \right. \\
&\quad \sum_{j < j' + 2} 2^{(\delta + pn - 2n)(j' - j)} 2^{-2pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p}) - 2j'p\beta} 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \\
&\quad \times \left. \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\}.
\end{aligned}$$

Applying Hölder's inequality once again yields

$$\begin{aligned}
II_{Q_r}^5 &\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left\{ \sum_{j < j' + 2} 2^{(j-j')[p(2n+2p(\gamma_2+1-\beta))]} \right. \\
&\quad \left. 2^{(\delta + pn - 2n)(j' - j)} 2^{pj'(\frac{n}{2} + \gamma_1 - \frac{n}{p})} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \sum_{w \in \mathbb{Z}^n} |Q_r|^{\frac{q\gamma_2}{n} - \frac{q}{p}} (1 + |w|)^{-N} \sum_{j \geq -\log_2 r} \left( \sum_{j < j' + 2} 2^{(j-j')[p(\gamma_1 + \gamma_2 + 1) - \delta]} \right)^{\frac{q-p}{p}} \\
&\quad \times \left\{ \sum_{j < j' + 2} 2^{(j-j')[p(\gamma_1 + \gamma_2 + 1) - \delta]} \left( \int_{2^{-2j'\beta}}^t \sum_{(\epsilon', k') \in S_r^{w, j'}} |u_{j', k'}^{\epsilon'}(s)|^p (s 2^{2j'\beta})^m \frac{ds}{s} \right)^{\frac{q}{p}} \right\} \\
&\lesssim \|u\|_{\mathbf{B}_{p,q,m}^{\gamma_1, \gamma_2, III}}.
\end{aligned}$$

## REFERENCES

- [1] D. Adams, *A note on Choquet integral with respect to Hausdorff capacity*, Function Spaces and Applications (Lund, 1986), 115-24, Lecture Notes in Math. **1302**, Springer, Berlin, 1988.
- [2] J. Alvarez, *Continuity of Calderón-Zygmund type operators on the predual of a Morrey space*, Clifford algebras in analysis and related topics (Fayetteville, AR, 1993), 309-319, Stud. Adv. Math., CRC, Boca Raton, FL, 1996.
- [3] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Mat. Iberoamericana, **13** (1997), 673-97.
- [4] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier-Stokes equations*, in: S. Friedlander, D. Serre (Eds.), Handbook of Mathematical Fluid Dynamics, vol. 3, Elsevier, 2004, pp. 161-44.
- [5] G. Dafni and J. Xiao, *Some new tent spaces and duality theorem for fractional Carleson measures and  $Q_\alpha(\mathbb{R}^n)$* , J. Funct. Anal., **208** (2004), 377-422.
- [6] G. Dafni and J. Xiao, *The dyadic structure and atomic decomposition of  $Q$  spaces in several real variables*, Tohoku Math. J., **57** (2005), 119-145.
- [7] M. Essen, S. Janson, L. Peng and J. Xiao,  *$Q$  spaces of several real variables*, Indiana Univ. Math. J., **49** (2000), 575-615.

- [8] C. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, The millennium prize problems, 57-67, Clay Math. Inst., Cambridge, MA.
- [9] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Reg. Conf. Ser. Math., vol. 79, Amer. Math. Soc., Providence, RI, 1991.
- [10] P. Germain, N. Pavlović and G. Staffilani, *Regularity of solutions to the Navier-Stokes equations evolving from small data in  $BMO^{-1}$* , Int. Math. Res. Not., **2007** (2007), doi:10.1093/imrn/rnm087.
- [11] Y. Giga and T. Miyakawa, *Navier-Stokes flow in  $\mathbb{R}^3$  with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations, **14** (1989), 577-618.
- [12] T. Kato, *Strong  $L^p$ -solutions of the Navier-Stokes in  $\mathbb{R}^n$  with applications to weak solutions*, Math. Z., **187** (1984), 471-480.
- [13] T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system*, Rend. Semin. Mat. Univ. Padova, **30** (1962), 243-260.
- [14] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **XLI** (1988), 891-907.
- [15] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157** (2001), 22-35.
- [16] Z. Lei and F. Lin, *Global mild solutions of Navier-Stokes equations*, Comm. Pure Appl. Math. **64**(2011), 1297-1304.
- [17] P. G. Lemarié-Rieusset, *Recent Development in the Navier-Stokes Problem*, Chapman & Hall/CRC Press, Boca Raton, 2002.
- [18] P. Li and Z. Zhai, *Well-posedness and regularity of generalized Navier-Stokes equations in some critical  $Q$ -spaces*, J. Functional Anal., **259** (2010), 2457-2519.
- [19] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, *New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets*, **18** (2012), 1067-1111.
- [20] C. Lin and Q. Yang, *Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier-Stokes equations*, J. Differential Equations, **254** (2013), 804-846.
- [21] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-linéaires*, Dunod/Gauthier. Villars, Paris, 1969 (in French).
- [22] C. Miao, B. Yuan and B. Zhang, *Well-posedness for the incompressible magneto-hydrodynamic system*, Math. Methods Appl. Sci., **30** (2007), 961-976.
- [23] Y. Meyer, *Ondelettes et Opérateurs, I et II*, Hermann, Paris, 1991-1992.
- [24] Y. Meyer and Q. Yang, *Continuity of Calder' on-Zygmund operators on Besov or Triebel-Lizorkin spaces*, Anal. Appl. (Singap.), **6** (2008), 51-81.
- [25] C. B. Morrey, *On the solutions of quas-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126-166.
- [26] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser., Duke Univ. Press, Durham, 1976.
- [27] L. Peng and Q. Yang, *Predual spaces for  $Q$  spaces*, Acta Math. Sci. Ser. B Engl. Ed., **29** (2009), 243-250.
- [28] Y. Sawano, *Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces*, Funct. Approx. Comment. Math., **38** (2008), 93-107.
- [29] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, London Mathematical Society Student Texts **37**, Cambridge University Press, 1997.
- [30] J. Wu, *Generalized MHD equations*, J. Differential Equations, 195 (2003), 284-312.

- [31] J. Wu *The generalized incompressible Navier-Stokes equations in Besov spaces*, Dyn. Partial Differ. Equ., **1** (2004), 381-400.
- [32] J. Wu, *Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces*, Comm. Math. Phys., **263** (2005), 803-831.
- [33] J. Wu, *Regularity criteria for the generalized MHD equations*, Comm. Partial Differential Equations, **33** (2008), 285-306.
- [34] Z. Wu and C. Xie,  *$Q$  spaces and Morrey spaces*, J. Functional Anal., **201** (2003), 282-297.
- [35] J. Xiao, *Homothetic variant of fractional Sobolev space with application to Navier-Stokes system*, Dyn. Partial Differ. Equ., **2** (2007), 227-245.
- [36] D. Yang and W. Yuan, *New Besov-type spaces and Triebel-Lizorkin-type spaces including  $Q$  spaces*, Math. Z. **265** (2010), 451-480.
- [37] Q. Yang, *Wavelet and Distribution*, Beijing Science and Technology Press, 2002.
- [38] Q. Yang, *Characterization of multiplier spaces with Daubechies wavelets*, Acta Math. Sci., **32** (2012), 2315-2321.
- [39] W. Yuan, W. Sickel and D. Yang, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics 2005 Editors: J.-M. Morel, Cachan F. Takens, Groningen B. Teissier, Paris.

DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU, GUANGDONG 515063, CHINA  
*E-mail address:* ptli@stu.edu.cn

DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND,  
 ST. JOHN'S, NL A1C, 5S7, CANADA  
*E-mail address:* jxiao@mun.ca

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, 430072, CHINA.  
*E-mail address:* qxyang@whu.edu.cn